LINEAR TIME QUADRATIC OPTIMAL CONTROL SYSTEMS

MSc PROJECT

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Linear Time Quadratic Optimal Control Systems

A Project submitted to the Department of Mathematics,
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In Partial Fulfillment of the Requirements for the Degree of
MASTER OF SCIENCE IN MATHEMATICS
(OPTIMIZATION)

Adane Akate

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We hereby certify that we have read and evaluated this project titled ‘‘**Linear Time Quadratic Optimal Control Systems**” prepared under our guidance by Adane Akate. We recommend that it be submitted as fulfilling the project requirement.

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As member of the board of Examiners of the MSc. Project Open Defense Examination, we certify that we have read and evaluated the project prepared by Adane Akate and examined the candidate. We recommend that the project be accepted as fulfilling the project requirement for the degree of *Master of Science in Mathematics* (Optimization).

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External Examiner	Signature	Date

Final approval and acceptance of the project is contingent up on the submission of its final copy to the Council of Graduate Studies (CGS) through the candidate’s school of graduate committee (SGC).

DEDICATION

To all my family members who have been a constant source of motivation, inspiration, and support.

STATEMENT OF THE AUTHOR

By my signature below, I declare that this project is my own work. I have followed all ethical and technical principles of scholarship in the preparation, and compilation of this project. Any scholarly matter that is included in the project has been given recognition through citation.

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ABBREVIATIONS

CSF	Chebyshev Scaling Function
ELE	Euler Lagrange Equation
HJBE	Hamilton-Jacobi-Bellman equation
LSF	Legendre Scaling Function
LTI	Linear Time Invariant
LTV	Linear Time Varying
OCPs	Optimal Control Problems
OMI	Operational Matrix of Integration
PMP	Pontryagin Minimum/Maximum Principle
QP	Quadratic Programming
TPBVP	Two-Point Boundary Value Problem

BIOGRAPHICAL SKETCH

The author was born on June 10 in 1993 in Amhara Regional State, East Gojam Zone, Gozamin Woreda. He attended his primary education at Gebela and Mutera Primary school. Then after, he joined Gozamin Secondary and Debre Markos Preparatory School to attend his secondary education. Then he joined Debre Tabor University in 2013 and received Bachelor of Science degree in Mathematics on July 4, 2016. He directly joined Postgraduate Program at Haramaya University, College of Natural and Computational Sciences, Department of Mathematics in 2016 to pursue a program of study for MSc. degree in Mathematics with specialization in Optimization.

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ABSTRACT

In this project, we discussed both linear time invariance and time varying optimal control problems with quadratic performance index, and approximated control variable, state variable and performance index using Legendre scaling function and Chebyshev scaling function methods with unknown coefficients. The linear time invariant and time varying problems were parameterized based on control-state parameterization technique such that the objective function and the constraints are casted in terms of state variable and control variable. These two methods were converting the linear time quadratic optimal control problems into quadratic programming problems and the converted problems were solved using MATLAB. Finally, some different illustrative examples were solved to show the applicability and effectiveness of the presented methods. Then these two methods were examined on the same examples and when we increase the order of polynomial (M), then the computational results of the proposed methods gave better results but when we compare these two methods, Legendre scaling function was better than Chebyshev scaling function with regard to optimal value. Hence, the Legendre scaling function method is more suitable for solving the linear time quadratic optimal control systems. This project can be extending to solve nonlinear time quadratic optimal control problems with inequality constraints using these two methods.

Keywords: Optimal control problems, state-control parameterization, Legendre scaling function, Chebyshev scaling function, operational matrix integration.

1. INTRODUCTION

1.1. Background of the Study

The optimal control systems developed by Pontryagin and his coworkers, is among the biggest successes in optimal control systems and published “On the theory of optimal processes” as the first paper on theory of optimal processes in late 1956. The optimal control can be deriving using Pontryagin’s maximum principle (a necessary condition also known as Pontryagin’s minimum principle or simply Pontryagin’s Principle). This principle used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state and/or input (controls) (Chachuat, 2007). Optimal control is a science that deals with linear and nonlinear optimal control problems (OCPs) and the main objective of optimal control is to determine control function that will the cause of a systems (plant) to satisfy the physical constraints and at the same time extremize (maximize or minimize) a performance criterion (performance index or cost function) (Wadi 2011).

Optimal control problem we mean a physical (or economic, or else) system whose behavior is described by a state variable and that can be controlled from outside by an input (or control) variable. The state variable (or function) is the set of variables (functions) used to describe the mathematical state of the system at a time t but according to Behroozifar *et al.* (2014) the value of state variable decreases when the time interval increases. The control variable (or function) is an operation that controls the recording, processing, or transmission of data at a time t but according to Fatihi *et al.* (2012) the value of control variable increases when the time interval increases in the case of linear time invariant OCPs. These two functions drive how the system works and how the desired control was founded (Rose, 2015). OCPs can be formulated discrete- time OCPs and continuous- time OCPs. Generally, the state variable and control variable are required to satisfy an ordinary differential equation (ODE) in the continuous time case or a difference equation (DE) in the discrete time case.

Definition of some key terms in optimal control problems:

- **Phases**: It is the position of an event in time. A problem may contain several phases, each with its own set of variables and equations.
- **Independent**: Variables that are independent (normally time).
- **State variable**: is one of the set of variables that are used to describe the mathematical of a dynamical system. States are continuous functions at time (independent) variables.
- **Final states:** Some of the states that are not eventually specifies at final time through which it is possible to modify the cost function.
- **Control variable:** It is a quantity or condition that is measured and controlled the systems.
- **Scalars:** A problem may also contain unknown scalar variables that are solve for at the same time as the control variable and state variable.
- **Parameters**: A constant numbers that are used to define a problem.

From a general perspective, an optimal control problem is an optimization problem. In optimal control theory, the optimizer is a function, not just a single value. The function that optimizes the performance index is called the optimal control (Rose, 2015).

Linear optimal control is a special sort of optimal control where the system that is controlled was assumed linear and the controller, the device that generates the optimal control is also constraints to be linear (Hans, 2007). Linear optimal control provides a framework for the unified study of the control problems studied via classical methods. It vastly extends the class of systems for which control designs are achieved (Anderson and Moore, 2007).

Polyaka and Tempob (2001) to study robust design of uncertain systems in a probabilistic setting applied linear quadratic optimal control. Systems affected by random bounded nonlinear uncertainty were considered so that classical optimization methods based on linear matrix inequalities. Bemporad *et al*. (2002) presented a technique to compute the explicit state-feedback solution to both the finite and infinite horizon linear quadratic optimal control problems subject to state and control (input) constraints. They succeeded in showing that the control law is piece-wise linear and continuous for both the finite horizon problem (model predictive control) and the usual infinite time measure (constrained linear quadratic optimal control).

Analytical solutions of OCPs are not always available, thus finding an approximate solution is at least the most logical way to solve them. By contrast, linear OCPs have readily computable solutions rather than nonlinear OCPs. OCPs with LTV systems are much more difficult to solve in comparison with LTI systems (Jain and Mittal, 2014). LTV systems have been frequently used to model systems that have non-stationary properties and undergo small magnitude there is vibration (Liu, 1997). The study of numerical methods has provided an attractive field for researchers of mathematical sciences which have rise to the appearance of different numerical computational methods and efficient algorithms to solve the OCPs (Kafash *et al.,* 2014). Numerical methods were applying many practical real world OCPs such as flight, water level in a tank, manufacturing systems, transportation etc., there are rigid requirements to be satisfied at every time point in the planning horizon. Such requirements are often express as continuous inequality constraints on the state and/or control (Li, 2011).

Generally, methods for solving OCPs can be classified as direct methods and indirect methods. Direct methods the dynamic OCPs converted into a QP problem and indirect methods are based on converting OCPs into a set of algebraic equations, then solving by ELE or HJBE (Abu Haya, 2011). Direct methods can be implemented using either discretization or parameterization techniques of the state and/or the control variables. Parameterization is the method that using polynomials in order to get the performance index and the constraints with the polynomials and the coefficients that make this representation valid (Wadi, 2011). The direct methods can be employed by using the parameterization technique which could be applied in three different ways: control parameterization, state parameterization and state-control parameterization (Jaddu, 2002). Here, we choose state-control parameterization technique in this project because there is no need to integrate the system of state equations as in control parameterization. In addition, the approximated optimal solutions of state variables and control variables are obtained at the same time.

We described state- control parameterization technique of the state variable, control variable at the same time and performance index using Legendre scaling function and Chebyshev scaling function methods with unknown coefficients. By these two methods, the control variable, state variable and performance index have obtained approximately and compared these two methods with regard to optimal value.

Razzaghi and Yousefi (2002) defined functions which they called Legendre wavelet-based method to solve the linear time-varying quadratic OCPs. However, these functions are not wavelets but scaling functions. The work of Jaddu (2006) presented the proof of multiplication of LSF as well as the multiplication operational matrix which was used in the development of a computational method to the linear time-varying quadratic OCPs. The method also developed by Wadi (2011) depends on state-control parameterization via Legendre scaling function and the iterative technique in which the nonlinear state equations were replacing by a sequence of LTV state equations. Ahmed (2017) has developed Legendre scaling function for solving OCPs converted into QP problem and the nonlinear optimal control problem into sequence of quadratic OCPs**.**

Shahrezaee (2016) was introduced numerical method for solving the OCPs. This method was used to a state parameterization technique which consider the state parameters of Chebyshev polynomials with unknown coefficients. In the work of Nagurka *et al*. (1991) Chebyshev-based representation of the state variable has been proposed for designing optimal control variables of unconstrained, linear, dynamic systems with quadratic performance indices. However, in this project including constrained, linear time optimal control problems with quadratic performance index has been considered.

The basic idea for solving linear time quadratic optimal control systems by using Legendre scaling function and Chebyshev scaling function methods are firstly, the systems changed into quadratic programming problems using these two methods, then the quadratic programming problems were solved by using MATLAB. To demonstrate applicability and effectiveness of the proposed methods, and various numerical examples were solved.

1.2. Statement of the Problems

Optimal control problem is an optimization problems finite-dimensional spaces (like in the case of discrete-time OCPs), or infinite-dimensional spaces (like in the case of a broad class of continuous-time OCPs).

The OMI and the Gauss method were utilized to reduce the optimal control problems to the algebraic equations (Razzaghi and Yousefi, 2002). Abu Haya (2011) discussed and solved LTI quadratic optimal control problems using Chebyshev wavelet. Elaydi and Abu Haya (2012) were solving both LTI and LTV optimal control problems using Chebyshev wavelet. Fatihi *et al.* (2012) finding the approximate solution of the optimal control of LTI systems using the Legendre wavelets and the operational matrix of stretch is derived and together with the OMI has been used to change the system of state equations into a set of algebraic equations. Fahroo and Ross (2002) presented the Chebyshev pseudospectral method for solving the OCPs with the state and control constraints. Jaddu (1998) has presented numerical methods to solve unconstrained and constrained linear OCPs and later, extended his ideas to solve nonlinear OCPs with terminal state constraints, control inequality constraints and simple bounds on state variables (Jaddu, 2002). Biswas *et al.* (2012) proposed a new algorithm to find the solution of optimal control for linear systems using Haar wavelet transformation.

In this project, we focused on special class of linear time invariant and time-varying OCPs, with quadratic objective functions and the constraints is linear. To find numerical solutions of linear time quadratic OCPs, how LSF and CSF with polynomial of degree three can be adapted and applied to these optimization problems but these two methods defined on the time interval $t \in [0,1]$. The general formulation of LTI quadratic optimal control problem as follows (Jaddu and Shimemura, 1999)

Find an optimal controller $u^*(t)$ and state variable $x^*(t)$ which minimizes the quadratic performance index.

$$\min\,(J(u)) = \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt \qquad (1.1)$$

Subject to

$$x'(t) = Ax(t) + Bu(t),\ \ x(t_0) = x_0, \qquad (1.2)$$

where $t \in [t_0, t_f], x_0 \in \mathbb{R}^n, x(t) \in \mathbb{R}^n$ is state variable, $u(t) \in \mathbb{R}^m$ is control variable, the vector, $x'(t)$ is the time derivative of the state of the system at time t. A, B are $n \times n$ and $n \times m$ time invariant matrix representation respectively. Q is an $n \times n$ positive semi definite matrix and R is an $m \times m$ positive definite matrix.

The general formulation of linear time varying quadratic optimal control problem as follows (Elaydi and Abu Haya, 2013)

Find an optimal controller $u^*(t)$ and state variable $x^*(t)$ that minimizes the following quadratic performance inde.

$$\min (J(u)) = \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{1.3}$$

Subject to

$$x'(t) = A(t)x(t) + B(t)u(t),\ x(t_0) = x_0 \tag{1.4}$$

where $t \in [t_0, t_f], x_0 \in \mathbb{R}^n, x(t) \in \mathbb{R}^n$ is state variable $u(t) \in \mathbb{R}^m$ is control variable, $x'(t)$ is the time derivative of the state of the system at time t, $A(t), B(t)$ are $n \times n$ and $n \times m$ time varying matrix representation respectively.

To perform this, it was tried to answer the project questions:

- How do we find the numerical solution of linear time quadratic optimal control problems?
- Which method is best (effective) from Legendre and Chebyshev scaling function for linear time quadratic optimal control systems with regard to optimal value?

1.3. Objectives

The general objective of this project was to solve linear time quadratic optimal control systems.

The study explored the following specific objectives:

- To use Legendre scaling function to approximate the state, control variables and performance index of both linear time invariant and time-varying quadratic OCPs.
- To use Chebyshev scaling function to approximate the state, control variables and performance index of both linear time invariant and time-varying quadratic OCPs.
- To compare Legendre scaling function and Chebyshev scaling function with regard to optimal value.

2. LITERATURE REVIEW

2.1. Optimal Control Problems

Optimal control is described the behavior of the system and maximizing or minimizing a given performance index (objective function) on the state variables and the control variables (Dolgy *et al*., 2017). The control variable represents the possibility of intervening in order to change the behavior of the system so that its performance index is optimized. A constraint on the state variable is the satisfaction of these constraints affect the evolution of the system, and restricts the admissible controls (Da Silva *et al.,* 2013). Swigart and Lall (2010) were developing controller synthesis algorithms for decentralized control problems, where individual subsystems were connecting over each other. Their work was focusing on the simplest information structure, consisting of two interconnected linear systems, and they constructed the optimal controller subject to a decentralization constraint via a spectral factorization approach. The state and/or control involved in the equation were approximating by finite terms of orthogonal series and using an OMI to eliminate the integral operations. The form of the OMI depends on the choice of the orthogonal functions like Legendry polynomials, Chebyshev polynomials and Wavelet functions (Tavallaei and Tousi, 2008).

Direct methods has been employed by direct substitution of the state variables and control variables into the performance index without constructing the Hamiltonian of the system as in indirect methods, the direct methods has been described as first parameterize then optimize objective function (Abu Haya, 2011). To solve OCPs using direct numerical methods for linear inequality constraints and equality constraints were depending on Legendre wavelets and Chebyshev wavelets (Razzaghi and Yousefi, 2002). Saberi *et al*. (2012) discussed analytical and approximate solution for the HJBE arising in OCPs using homotopy perturbation method (HPM). Kafash *et al.* (2014) presented a computational method based on state parameterization of state variable by using Boubaker polynomials for solving OCPs.

In solving QP problem, one approach is to treat the initial state conditions as constraints that are included in the performance index using a Lagrange multiplier technique. In this, approaches the necessary condition of optimality is derive as a system of linear algebraic equations from which near to control variable can be designed (Nagurka *et al.,* 1991). Lamperski and Cowan (2013) gave a solution to the linear quadratic regulator (control) problem in which the state is perfectly known, but the controller's measure of time is a stochastic process derived from a strictly increasing the process. They concluded that the optimal controller is linear and can be compute from a generalization of the classical Riccati differential equation.

Jaddu (2006) discussed the Chebyshev wavelet to solve the linear quadratic optimal control problem with terminal constraints. The method is based on converting the OCPs into mathematical programming and he used the operational matrix of differentiation. Babolian and Fattahzadeh (2007) extended this work to study the presented OMI of Chebyshev wavelets basis and the product of operational matrix. Here we presented a scaling function based numerical methods to solve a linear time quadratic OCPs. The methods are LSF and CSF to approximate the state variables, control variables and performance index. Therefore, using these methods the OCPs were transforming into a QP problem. In the proposed methods, the main object is to reduce the system of dimensions to achieve a robust and easy to approximate the solution.

Razzaghi and Yousefi (2002) studied the numerical method for solving linear OCPs with inequality constraints. The method consists of reducing the OCPs to a set of algebraic equations by expanding the state variable $x(t)$ and control variable $u(t)$ using Legendre wavelets with unknown coefficients. Elaydi and Abu Haya (2012) also studied the systems of state variables and control variables using a finite length Chebyshev wavelet. The aim of this method is the determination of the optimal control and state variables by a direct method of solution based on Chebyshev wavelet. Ghasemi and Tavassoli (2011) presented a solution of time-varying systems by using Chebyshev wavelets.

Balachandran and Murugesan (1992) studied a single-term Walsh series (STWS) method applied on linear optimal control of singular systems. Park *et al*. (2004) used RK-Butcher algorithm to compute numerical solution of LTI optimal control of singular systems. Behroozifar *et al.* (2014) extended this work to constrained OCPs of time-varying singular systems with a quadratic cost function using Bernstein polynomials. The minimization of the cost function for a predicted time invariant and time varying systems are achieved by considering the prediction horizon as a combination of infinite and finite horizon parts. The number of future prediction steps depends upon the problem and fixed variable chosen during the controller design (Dutka and Grimble, 2004). Wadi (2011) investigated OCPs with LTI, LTV systems, and nonlinear systems. LTI and LTV systems are the class of dynamic systems exploited in both theory and practice. Since the most of real processes record some of LTI and LTV systems usually serve as an approximation of the processes encountered in real life.

2.2. Linear Quadratic Optimal Control Systems

Linear quadratic optimal control systems is a special type of optimal control that deals with linear systems (in state and in control) and minimization of objective or cost function that are quadratic performance index (Hans, 2007). In Rantzer and Johansson (2000) the use of piecewise quadratic cost functions was extended from stability analysis of piecewise linear systems to performance analysis and the optimal control cost are obtained by semidefinite programming based on the Bellman inequality and gave an approximation of optimal control variable. By approximating each state variable by a finite-term, shifted Chebyshev series, the linear quadratic OCPs could be casted as a QP problem (Nagurka *et al.*, 1991). Zeid and Yousefi (2016) developed the Adomian decomposition method (ADM) to approximate the solution of fractional optimal control problems (FOCPs) where the dynamic of system is a linear control system with constant coefficient and the cost function is defined in a quadratic form.

Jaddu and Shimemura (1999) proposed a method to solve the linear-quadratic and the nonlinear OCPs by using Chebyshev polynomials to parameterize some of the state variables, and then the remaining state variables and the control variables are determined from the state equations. Yen and Nagurka (1992) proposed a method based on the state parameterization using Fourier series, to solve the linear-quadratic OCPs (with equal number of state variables and control variables) subject to state and control inequality constraints. In addition, Razzaghi and Elnagar (1994) proposed a method to solve the unconstrained linear-quadratic optimal control problem with equal number of state variables and control variables. Their approaches are based on using the shifted Legendre polynomials to parameterize the derivative of each of the state variables.

Continuous LTI or LTV systems is one which deals with continuous time signals (function) and satisfies both the principles of linearity time invariance or time varying. For continuous-time systems, the minimum-time problem is generally addressed by applying the minimum or maximum principle of Pontryagin (Bako *et al.,* 2011). The Pontryagin's Minimum Principle is applied to find optimal switching level applying minimum-time optimal control rules (Shariatmadar and Jafarian, 2017). Jiang, Y. and Jiang, Z. (2012) presented an iteration policy approach for finding the approximate optimal controllers for continuous-time linear systems with completely unknown dynamic system. Continuous-time (CT) signals (function) are defined for every value of time on an interval (possibly, an infinite interval) and take on values in an interval.

Discrete-Time (DT) signals (functions) are defined only at integer values of time. Borrelli *et al*. (2005) study the solution of OCPs for constrained discrete-time linear systems based on quadratic performance index. Huang *et al.* (2008) has been studied infinite horizon discrete-time linear quadratic optimal control with both state variable and control variable. For discrete-time uncertain linear systems with constraints on inputs (control) and states, this develop an approach to determine state feedback controllers based on a min–max for both open-loop and closed-loop control formulation (Bemporad *et al.*, 2003). In fact, discrete-time problems are more general than typical continuous-time problems in one respect the dimensionality of the state vectors and control vectors can vary with time (Rockafellar and Wets, 1990). Teo *et al*. (1990) discussed a discrete time OCPs can be solved as a mathematical programming problem by regarding all the states variables and the control variables as decision variables and the difference equations as algebraic constraints.

3. MATERIALS AND METHODS

Sources in the web site and libraries were used to collect all the pieces of information about both linear time invariant and time varying quadratic optimal control systems and recorded subsequently.

Specifically,

- Books, Relevant journals, published articles, related studies from internet services and any available material that support the study were consulted to gather information about the linear time quadratic optimal control systems.
- The collected information was analyzed and arranged keeping coherence.
- Important concepts, definitions, and examples were discussed to make ideas clear.
- Using Legendre scaling function both linear time invariant and time varying quadratic optimal control problems were converted into quadratic programming problems.
- Using Chebyshev scaling function both linear time invariant and time varying quadratic optimal control problems were converted into quadratic programming problems.
- The converted quadratic programming problems were solved using MATLAB.

4. PRELIMINARIES

In this chapter, we deal with definitions and concepts which are important for the study of both solving linear time invariance and time varying quadratic optimal control systems.

4.1. Control Systems

Control system is the system that includes at least the plant (system) and its controller. It may include other subsystems and components (e.g., sensors, signal conditioning and modification). Controlling a system or an object means to influence the behavior of the system or an object to achieve a desired goal. Controlled variable is the quantity or condition that is measured and controlled the systems (De Silva, 2009).

The essential components of a control system are:

1. The plant (system) which is to be controlled;
2. One or more sensors, which give information about the plant (system); and
3. The controller, the "heart" of the control system, which compares the measured values to their desired values and adjusts the input variables to the plant (system).

Basically, there are two types of control systems: the open loop control system and the closed loop control system. They can be both represented by block diagrams. A block diagram used to represent processes, while arrows are used to connect different input, process and output parts.

4.1.1. Open Loop Control System

Definition 4.1. (Open-Loop). If the optimal control law is determined as a function of time for a specified initial state value, that is, $u^*(t) = w(t, x(t_0))$ then the optimal control is said to be in open-loop form. The input and output of an open loop system are unrelated. An open-loop control system does not measure the output to determine the control action; its accuracy depends on its calibration, and thus it is only used when there are no perturbations and the system is perfectly known. The drawback of an open loop control system is that it is incapable of making automatic adjustments. Even when the magnitude of the output is too big or too small, the system will not make the appropriate adjustments. For this reason, an open loop control system is not suitable for use as a complex control system. For example, when a washing machine finishes cleaning the clothes, the user will need to check whether the clothes are clean or not; if they are not, they have to be put back into the machine and washed again (Chachuat, 2007).

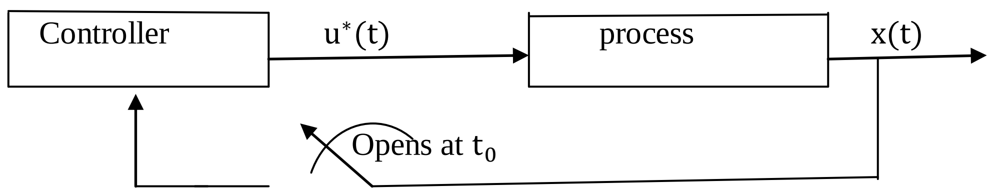

Figure 1. Open loop control system

4.1.2. Closed Loop Control System

Definition 4.2. (Closed-Loop). If a functional relation of the form $u^*(t) = w\,(t, x(t))$ can be found for the optimal control at time t, then w is called a closed-loop optimal control for the problem. The terms optimal feedback control or optimal control law are also often used. Sometimes, we may use the output of the control system to adjust the input signal is called feedback. Feedback control is a central to managing computing systems and networks. A closed loop control system compares the output with the expected result or command status, and then it takes appropriate control actions to adjust the input signal. The output signal is feedback to the input to produce a new output. A well-designed feedback system can often increase the accuracy of the output (Chachuat, 2007).

One advantage of using the closed loop control system is able to adjust its output automatically by feeding the output signal back to the input. When the load changes, the error signals generated by the system will adjust the output. However, closed loop control systems are generally more complicated and thus more expensive to make.

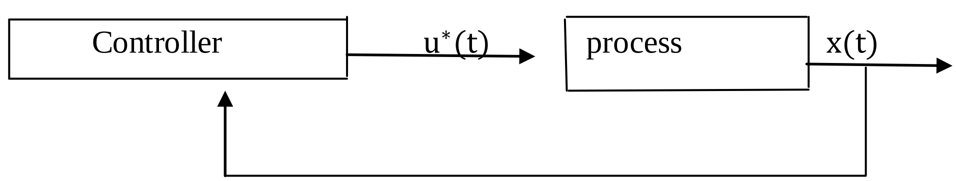

Figure 2. Closed Loop Control Systems

First-order Necessary conditions: Consider equation (1.1) and (1.2) with fixed endpoints $t_0 \leq t_f$ where the objective function (l) is quadratic function and the constraint (f) is linear function in (t, x, u) and have continuous first partial derivatives with respect to x and u. Suppose that u^* is a local minimizer for the problem, and let x^* denote the corresponding response. Then, there is a function u^*, x^* and λ^* those satisfy the system

ODE: $x' = H_\lambda$, $x(t_0) = x_0$ (Given condition)

(ADJ): $\lambda' = -H_x$

(M): $H(u^*) = \min_u H(t, x^*, u, \lambda^*)$ (u^* minimizes H with respect to u at x^* and λ^*)

(T): $\lambda(t_f) = 0$

where $H(t, x, u, \lambda) = l + \lambda f$ is Hamiltonian function

l= objective function, f= constraint

These equations are known collectively as the Euler-Lagrange equations (ELE).

4.2. Methods for Solving the Optimal Control Problems

4.2.1. Direct Methods

The direct methods are based on obtaining the solution through a direct minimization of the performance index, subject to constraints, of the OCPs (Ahmed, 2017). Direct methods are based on transforming the optimal control problems into mathematical programming problems. Direct methods can be implemented by using either discretization or parameterization methods (Abu Haya, 2011). In this project, we focused parameterization technique to parameterize state variable, control variable, and then linear time quadratic optimal control problems converted into QP problems, which are much easier than the original problems or in order to solve QP problems using MATLAB code.

4.2.1.1. Discretization

Discretization is a process in which the time interval $t \in [t_0, t_f]$ is to be divided into an equal time segments,

$$t_0 \leq t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_i = t_f \tag{4.1}$$

As a result, depending on the discretization technique, the variable(s) is (are) sampled at each time point in (4.1). Basically, there are two discretization technique used in optimal control problem: Control-state discretization and control discretization (Abu Haya, 2011).

1. Control-State discretization

Apply this method to discretize both control variables $u(t)$ and state variables $x(t)$. As a result, the vector which contains a sequence of unknown state variables and control variables, will be produced.

2. Control discretization

This approach is to discretize the control variables $u(t)$ only. In order to get the state variables $x(t)$, it is necessary to integrate the system of state equations. This would produce state variables that are a function of the control variables. An advantage of this technique over control-state discretization is that the resulted system is lower in dimension.

4.2.1.2. Parameterization

Parameterization technique is an essential part of this project; it is a process in which a functions or variables are approximated using known functions with unknown coefficient. Parameterization can be employed by one of the three forms: Control parameterization, control-state parameterization and state parameterization. In this project, we used control-state parameterization technique.

1. Control Parameterization

In this technique, only the control variables $u(t)$ are approximated by a finite length series of known function with unknown coefficient.

2. State Parameterization

In this technique, only the state variables $x(t)$ are approximated by a finite length series of known functions with unknown coefficient.

3. Control-State Parameterization

Using this technique, both control variables $u(t)$ and state variables $x(t)$ are approximated by a finite length series of known function with unknown coefficient.

4.2.2. Indirect Methods

Indirect method transforms the problem into another form before solving it. The indirect method is sometimes described as first optimize then discretize. This is because optimality conditions are found before numerical techniques are introduced. Indirect methods are based on solutions that satisfy the HJBE or on solutions that convert the optimal control problem into a TPBVP. This approach is based on finding an approximate solution to the HJBE or the nonlinear TPBVP by using ELE. The approximated feedback control law obtained by this technique is solved successively (Ahmed, 2017).

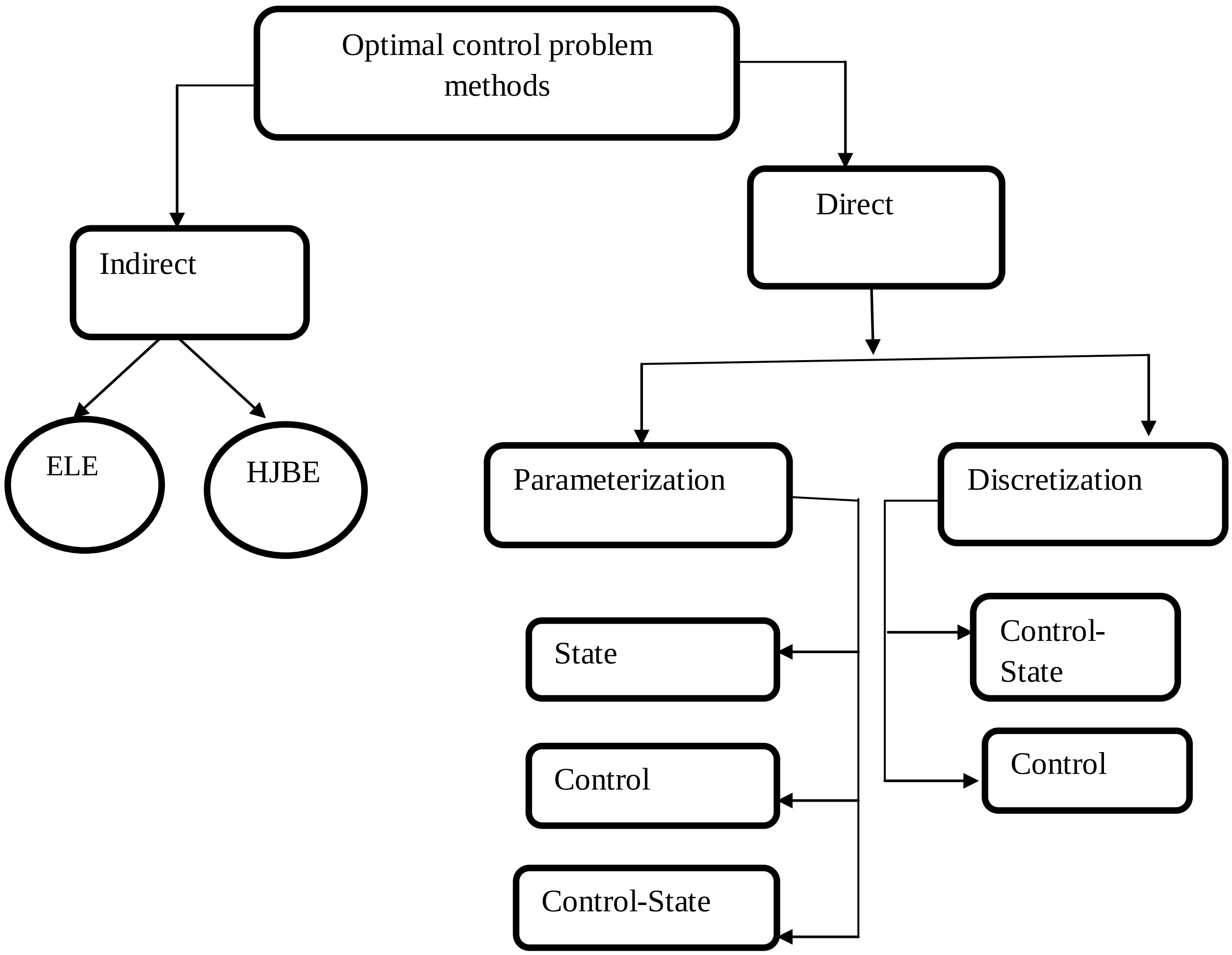

Figure 3. Computation procedures of optimal control problems

4.3. Kronecker Product

In mathematics, the Kronecker product, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix. It is a generalization of the outer product (which is denoted by the same symbol) from vectors to matrices, and gives the matrix of the product with respect to a standard choice of basis. The Kronecker product should not be confused with the usual matrix multiplication, which is an entirely different operation (Langville *et al.*, 2004).

Definition 4.3: If A is a $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & & a_{2n}B \\ \vdots & & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

4.4. Legendre Scaling Function

In this section, we will introduce the definition, basis of Legendre scaling function. This basis will be the basic of this project in the following section. Legendre scaling function is used to approximate both state variables and control variables because the optimal solutions of state variables and control variables are a function of time so it is better to use this function for quadratic optimal control problems. Legendre scaling function can be defined as follows

$$\Phi_{nm}(t) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{K/2} P_m(2^K t - 2n + 1)\,, & \text{for } \frac{2n-2}{2^K} \le t \le \frac{2n}{2^K} \\ 0\,, & \text{Otherwise} \end{cases} \tag{4.2}$$

where P_m is the Legendre polynomial of order m; n refers to the section of time interval, $n = 1, 2, \dots, 2^{K-1}$; K is the scaling parameter and can assume any positive integer and $t \in [0,1]$. From this the Legendre polynomial can be given

$$P_m(t) = \frac{1}{2^m m!} \frac{d^m}{dt^m} (t^2 - 1)^m \tag{4.2a}$$

$$P_0(t) = 1$$

$$P_1(t) = t$$

$$P_2(t) = \frac{1}{2}(3t^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5t^3 - 3t)$$

$$\vdots$$

Now by using equations (4.2) and (4.2a) and by choosing $M = 3$ and $K = 2$, the four basis Legendre scaling functions can be given for n = 1

$$\left.\begin{aligned} \Phi_{10}(t) &= \sqrt{2} \\ \Phi_{11}(t) &= \sqrt{6}(4t-1) \\ \Phi_{12}(t) &= \sqrt{\frac{5}{2}}[3(4t-1)^2-1] \\ \Phi_{13}(t) &= \sqrt{\frac{7}{2}}[5(4t-1)^3-3(4t-1)] \end{aligned}\right\} 0 \le t < \frac{1}{2} \tag{4.2b}$$

for n = 2

$$\left.\begin{aligned} \Phi_{20}(t) &= \sqrt{2} \\ \Phi_{21}(t) &= \sqrt{6}(4t-3) \\ \Phi_{22}(t) &= \sqrt{\frac{5}{2}}[3(4t-3)^2-1] \\ \Phi_{23}(t) &= \sqrt{\frac{7}{2}}[5(4t-3)^3-3(4t-3)] \end{aligned}\right\} \frac{1}{2} \le t \le 1 \tag{4.2c}$$

4.4.1. Operational Matrix of Integration (OMI) for Legendre Scaling Function

Definition 4.4 Let P be an operational matrix of integration which is obtained from the integral of Legendre scaling function from 0 to t and these matrices play an important role to modeling the problems. According to Razzaghi and Yousefi (2002) this is used to change the system of state equations into a set of algebraic equations which can be solved using software. This is represented as follows:

$$\int_0^t \Phi(\tau)d\tau = P\Phi(t)$$

where $\Phi(t) = [\Phi_{10}(t)\ \ \Phi_{11}(t) \dots \Phi_{1M}(t)\ \ \Phi_{20}(t) \dots \Phi_{2M}\ \ \Phi_{2^{K-1}0} \dots \Phi_{2^{K-1}M}(t)]^T$.

and P is called the cofficent matrix of $\Phi(t)$

$$P = \frac{1}{2^K}\begin{bmatrix} D & U \\ O & D \end{bmatrix} \tag{4.3}$$

where P is a $(2^{K-1}(M+1) \times 2^{K-1}(M+1))$ operational matrix of integration and O, D, U are $(M+1) \times (M+1)$ matrices and given by

$$U = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{35}} & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \frac{-1}{\sqrt{4M^2-16M+15}} & 0 & \frac{1}{\sqrt{4M^2-8M+3}} \\ & & & & & 0 & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{4M^2-8M+3}} & 0 \end{bmatrix}$$

O is matrix that all entire elements are zeros.

Note: The integration of multiplication of Legendre scaling function and its transpose in the interval $t \in [0,1]$ is equal to identity matrix since Legendre scaling functions are orthonormal is as follows.

$$\int_0^1 \Phi(t)\Phi^T(t)\,dt = I_N \tag{4.4}$$

where I_N, is identity matrix of dimension N, $(N = 2^{K-1}(M+1))$.

4.4.2. Continuity Test of Legendre Scaling Function

Scaling functions are not supported on whole interval $a \le x < b$; so these functions divide the interval of interest to number of sections depending on the value of scaling parameter K; for this reasons we have to add additional constraints to ensure the continuity of the state variables between different sections. There are $2^{K-1} - 1$ points at which the continuity of state variables.

$$t_i = \frac{i}{2^{K-1}} \quad , i = 1, 2, \dots, 2^{K-1} - 1 \tag{4.5}$$

In addition, there is dimension of equality constraints is $(2^{K-1} - 1)s$ can be given as follows

$$(I_s \otimes \Phi')a = o_{(2^{K-1}-1)_{s\times 1}} \tag{4.6}$$

where Φ' is continuity matrix

However, the dimension of matrix of continuity ensured constraints is $(2^{K-1}-1)\times(2^{K-1}(M+1))$.

4.4.3. Quadratic Programming Problem for Legendre Scaling Function

Quadratic programs are a family of problems solved in optimization applications. The goal is to find an optimal solution minimum or maximum of an objective function with quadratic, linear and constant terms.

In this section, we try to construct the quadratic form of optimal control problem can be easily solved using MATLAB code.

From the integration of state equation (1.2) can be rewritten using operational matrix of integration and unknown coefficients in the following form.

$$[(A\otimes P^T)-I_{Ns} \quad (B\otimes P^T)]\begin{bmatrix} a \\ b \end{bmatrix} = -\xi_0\delta \tag{4.7}$$

where $\delta=\frac{\sqrt{2}}{2^{K/2}}$, A is an $n\times n$ matrix and B is an $n\times m$ are the coefficient matrices of state variable and control variable from the equation (1.2) respectively, ξ_0 is the initial column vector with the dimension of $2^{K-1}(M+1)$, P is operational matrix of integration, I_{Ns} is identity matrix with the dimension of $2^{K-1}(M+1)$ and a, b are unknown coefficients.

By combining equations (4.6) and (4.7), we get the following form of equality constraints.

$$\begin{bmatrix} (A\otimes P^T)-I_{Ns} & (B\otimes P^T) \\ I_s\otimes\Phi' & 0_{(2^{K-1}-1)_{s\times Nr}} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\xi_0\delta \\ 0_{(2^{K-1}-1)_{s\times 1}} \end{bmatrix} \tag{4.8}$$

Then compact quadratic form is as follows:

$$\left.\begin{array}{rl} \min_z & z^T H z \\ \text{Subject to} & Fz = h \end{array}\right\} \tag{4.9}$$

$$\text{where,} \quad z = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$H = \begin{bmatrix} Q\otimes I_N & O_{Ns\times Nr} \\ O_{Nr\times Ns} & R\otimes I_N \end{bmatrix}$$

$$F = \begin{bmatrix} (A\otimes P^T)-I_{Ns} & (B\otimes P^T) \\ (I_s\otimes\Phi') & 0_{(2^{K-1}-1)_{s\times Nr}} \end{bmatrix}$$

$$h = \begin{bmatrix} -\xi_0 \delta \\ 0_{(2^{K-1}-1)_{s\times 1}} \end{bmatrix}$$

4.5. Chebyshev Scaling Function

We will introduce the definition, basis of Chebyshev scaling function. This basis will be the basic of this project in the following section. Chebyshev scaling function is used to for approximate both state variables and control variables because the optimal solution of state variables and control variables are a function of time so it is better to use this function for quadratic optimal control problems and Chebyshev scaling function can be defined is as follows

$$\Psi_{nm}(t) = \begin{cases} \dfrac{\alpha_m 2^{(k-1)/2}}{\sqrt{\pi}} T_m(2^k t - 2n + 1), \dfrac{n-1}{2^{k-1}} \le t \le \dfrac{n}{2^{k-1}} \\ \qquad 0\,, \quad \text{otherwise} \end{cases} \tag{4.10}$$

where $\alpha_m = \begin{cases} \sqrt{2} & , \ m = 0 \\ 2 & , \ m = 1,2,3,\dots \end{cases}$

Here, $T_m(t)$ are the Chebyshev polynomials of order m,

$$T_0(t) = 1,$$

$$T_1(t) = t,$$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 3t$$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t),\ m = 1, 2, 3, \dots. \tag{4.11a}$$

Now by using equations (4.10) and (4.11a) and by choosing $M = 3$ and $K = 2$, then the four basis Chebyshev scaling functions can be given For $n = 1$

$$\left.\begin{aligned} \Psi_{10}(t) &= \frac{2}{\sqrt{\pi}} \\ \Psi_{11}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(4t-1) \\ \Psi_{12}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(2(4t-1)^2-1) \\ \Psi_{13}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(4(4t-1)^3-3(4t-1)) \end{aligned}\right\}, 0 \le t < \frac{1}{2} \tag{4.11b}$$

For n=2

$$\left.\begin{aligned} \Psi_{20}(t) &= \frac{2}{\sqrt{\pi}} \\ \Psi_{21}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(4t-3) \\ \Psi_{22}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(2(4t-3)^2-1) \\ \Psi_{23}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}\left(4(4t-3)^3-3(4t-3)\right) \end{aligned}\right\} \frac{1}{2} \le t \le 1 \tag{4.11c}$$

4.5.1. Operational Matrix of Integration (OMI) for Chebyshev Scaling Function

Definition 4.5 Let P be an operational matrix of integration which is obtained from the integral of Chebyshev scaling function from 0 to t and these matrices play an important role to modeling the problems. According to Ghasemi and Tavassoli (2011), this matrix is used to change the system of state equations into a set of algebraic equations which can be solved using software. This is represented as follows:

$$\int_0^t \Psi(\tau)d\tau = P\Psi(t)$$

$$\Psi(t) = [\Psi_{1m}(t), \Psi_{2m}(t), \Psi_{3m}(t), \dots, \Psi_{2^{k-1}m}(t)]^T$$

So the matrix P is the cofficent of $\Psi(t)$

$$P = \begin{bmatrix} C & S \\ O & C \end{bmatrix} \tag{4.12}$$

where P is a $(2^{K-1}(M+1) \times 2^{K-1}(M+1))$ operational matrix of integration and O, C, S are $(M+1) \times (M+1)$ matrices and give

$$S = \frac{\sqrt{2}}{2^{K-1}} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{15} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{-1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$C=\frac{1}{2^{k-1}}\begin{bmatrix}\frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0\\ \frac{-1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0\\ \frac{-1}{3\sqrt{2}} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0\\ \frac{-1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{-1}{4(M-1)}\\ \frac{-1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0\end{bmatrix}$$

O is matrix that all entire elements are zeros

Note: The integration of the product of two Chebyshev scaling function vectors is obtained as for $k=1,2,3,\ldots$ and for M=3

$$\int_0^1 \Psi(t)\Psi^T(t)\,dt = RR \tag{4.13}$$

$$\text{where,}\qquad RR=\begin{bmatrix}G & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & G\end{bmatrix}$$

$$G=\begin{bmatrix}\frac{2}{\pi} & 0 & \frac{-2\sqrt{2}}{3\pi}\\ 0 & \frac{4}{3\pi} & 0\\ \frac{-2\sqrt{2}}{3\pi} & 0 & 0.549\end{bmatrix}$$

4.5.2. Continuity Test of Chebyshev Scaling Function

To insure the continuity of the state variables between the different sections so we must add constraints. There are $2^{k-1}-1$ points at which the continuity of the state variables has to ensure. These points are

$$t_i=\frac{i}{2^{k-1}} \quad ,i=1,2,\ldots,2^{K-1}-1 \tag{4.14}$$

In addition, there is $(2^{K-1}-1)s$ equality constraints can be given as follows

$$(I_s\otimes\Phi')a=o_{(2^{K-1}-1)_{s\times1}} \tag{4.15}$$

However, the dimension of continuity matrix Φ' or ensured constraints is $(2^{K-1}-1)\times 2^{K-1}(M+1)$.

4.5.3. Quadratic Programming Problem for Chebyshev Scaling Function

In this section, we try to construct the quadratic form of optimal control problem can be easily solved using MATLAB code.

From the integration of state equation (1.2) can be rewritten using operational matrix of integration and unknown coefficients in the following form.

$$[(A\otimes P^T) - I_{Ns} \quad (B\otimes P^T)]\begin{bmatrix} a \\ b \end{bmatrix} = -g_0\delta \tag{4.16}$$

where $\delta = \frac{\sqrt{\pi}}{2}$, A is an $n\times n$ matrix and B is an $n\times m$ are the coefficient matrices of state variable and control variable from the equation (1.2) respectively, g_0 is the initial column vector, P is operational matrix of integration with the dimension of $2^{K-1}(M+1)$, I_{Ns} is identity matrix with the dimension of $2^{K-1}(M+1)$ and a, b are unknown coefficients.

By combining equations (4.15) and (4.16), we get the following form of equality constraints

$$\begin{bmatrix} (A\otimes P^T) - I_{Ns} & (B\otimes P^T) \\ I_s\otimes\Phi' & o_{(2^{K-1}-1)_{s\times Nr}} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -g_0\delta \\ o_{(2^{K-1}-1)_{s\times 1}} \end{bmatrix} \tag{4.17}$$

Then compact quadratic form is as follows:

$$\left.\begin{array}{ll} \min_z & z^T H z \\ \text{Subject to} & Fz = h \end{array}\right\} \tag{4.19}$$

where, $z = \begin{bmatrix} a \\ b \end{bmatrix}$

$$H = \begin{bmatrix} RR\otimes Q & O_{Ns\times Nr} \\ O_{Nr\times Ns} & RR\otimes R \end{bmatrix}$$

$$F = \begin{bmatrix} (A\otimes P^T) - I_{Ns} & (B\otimes P^T) \\ (I_s\otimes\Phi') & o_{(2^{K-1}-1)_{s\times Nr}} \end{bmatrix}$$

$$h = \begin{bmatrix} -g_0\delta \\ o_{(2^{K-1}-1)_{s\times 1}} \end{bmatrix}$$

5. LINEAR TIME QUADRATIC OPTIMAL CONTROL PROBLEMS

5.1. LTI Quadratic Optimal Control Problem

Before approximating the states variable and control variable, it is necessary to transform the time in the optimal control problem $t \in [t_0, t_f]$ into the interval $\tau \in [0,1]$; because Legendre scaling function (Elaydi *et al*., 2012) and Chebyshev scaling function (Fatihi *et al*., 2012) are defined on the interval $\tau \in [0,1]$.

Then we can obtain by using

$$\tau = \frac{t}{t_f} \qquad \text{so} \qquad dt = t_f d\tau$$

Then the transformed optimal control problem is

$$\min J = t_f \int_0^1 (x^T(t)Qx(t) + u^T(t)Ru(t))d\tau \tag{5.1}$$

$$\text{subject to} \quad \frac{dx(t)}{d\tau} = t_f(Ax(t) + Bu(t))$$

5.1.1. Approximation of State and Control Variable Using Legendre Scaling Function

The approximation of state variables and control variables for the problem of linear time invariance quadratic optimal control problems or equation (1.1) with linear constraint or equation (1.2) using Legendre scaling function is as follows

$$x_i(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} a^i{}_{nm} \Phi_{nm}(t) \ , \quad \text{where} \quad i = 1, 2, 3\ , \ldots, s$$

$$u_i(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} b^i{}_{nm} \Phi_{nm}(t) \ , \qquad \text{where} \quad i = 1, 2, 3, \ldots, r$$

These equations can be expanding to

$$x(t) = a_{10}\ \Phi_{10}(t) + a_{11}\ \Phi_{11}(t) + \cdots + a_{1M}\ \Phi_{1M}(t) \ldots + \cdots a_{2^{K-1}0}\Phi_{2^{K-1}0}(t) + \cdots + a_{2^{K-1}M}\Phi_{2^{K-1}M}(t)$$

$$u(t) = b_{10}\ \Phi_{10}(t) + b_{11}\ \Phi_{11}(t) + \cdots + b_{1M}\ \Phi_{1M}(t) \ldots + \cdots b_{2^{K-1}0}\Phi_{2^{K-1}0}(t) + \cdots + b_{2^{K-1}M}\Phi_{2^{K-1}M}(t)$$

These equations can be written in compact form are

$$\left.\begin{aligned} x(t) &= \left(I_s \otimes \Phi^T(t)\right)a \\ u(t) &= \left(I_r \otimes \Phi^T(t)\right)b \end{aligned}\right\} \tag{5.2}$$

where I_s and I_r are $s \times s$ and $r \times r$ identity matrices respectively and $\Phi(t)$is $N \times 1$, $N = 2^{K-1}(M+1)$

Vector of Legendre scaling function and unknown coefficients are given by:

$$\Phi(t) = [\Phi_{10}(t) \quad \Phi_{11}(t) \ldots \Phi_{1M}(t) \quad \Phi_{20}(t) \ldots \Phi_{2M} \ldots \quad \Phi_{2^{K-1}0}(t) \ldots \Phi_{2^{K-1}M}(t)]^T$$

$$a = [a_{10} \; a_{11} \; a_{12} \; \ldots \; a_{1M} \, a_{20} \; \ldots a_{2M} \ldots \; a_{2^{K-1}0} \; \ldots \; a_{2^{K-1}M}]^T$$

and $$b = [b_{10} \; b_{11} \; b_{12} \; \ldots \; b_{1M} \, b_{20} \; \ldots b_{2M} \ldots \; b_{2^{K-1}0} \; \ldots \; b_{2^{K-1}M}]^T$$

where a and b are vectors of unknown coefficients of dimension $sN \times 1$ and $rN \times 1$.

5.1.2. Approximation of State and Control Variable Using Chebyshev Scaling Function

The basic idea of Chebyshev scaling function is to approximate the state and control variables for the equation (1.1) by using a finite series is as follows.

$$x_i(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} a^i{}_{nm} \Psi_{nm}(t) \;, \qquad i = 1, 2, 3, \ldots, s$$

$$u_i(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} b^i{}_{nm} \Psi_{nm}(t) \,, \qquad i = 1, 2, 3, \ldots, r$$

These equations can be expanding to

$$\begin{aligned} x(t) &= a_{10}\; \Psi_{10}(t) + a_{11}\; \Psi_{11}(t) + \cdots + a_{1M}\; \Psi_{1M}(t) \ldots + \cdots\; a_{2^{K-1}0} \Psi_{2^{K-1}0}(t) \\ &\quad + \cdots + a_{2^{K-1}M} \Psi_{2^{K-1}M}(t) \\ u(t) &= b_{10}\; \Psi_{10}(t) + b_{11}\; \Psi_{11}(t) + \cdots + b_{1M}\; \Psi_{1M}(t) \ldots + \cdots\; b_{2^{K-1}0} \Psi_{2^{K-1}0}(t) \\ &\quad + \cdots + b_{2^{K-1}M} \Psi_{2^{K-1}M}(t) \end{aligned}$$

We can write these two equations in compact form as:

$$\left.\begin{aligned} x(t) &= \left(I_s \otimes \Psi^T(t)\right)a \\ u(t) &= \left(I_r \otimes \Psi^T(t)\right)b \end{aligned}\right\} \tag{5.3}$$

where $I_s$and I_r are $s \times s$ and $r \times r$ identity matrices and $\Psi(t)$is $N \times 1, N = 2^{k-1}(M+1)$.

Vector of chebyshev scaling functions and unknown coefficints are given by:

$$\Psi(t) = [\Psi_{10}(t) \quad \Psi_{11}(t) \ldots \Psi_{1M}(t) \quad \Psi_{20}(t) \ldots \Psi_{2M} \; \ldots \; \Psi_{2^{K-1}0}(t) \ldots \Psi_{2^{K-1}M}(t)]^T$$

$$a = [a_{10}\ a_{11}\ \ a_{12}\ \ \dots\ a_{1M}\, a_{20}\ \ \dots a_{2M}\ \dots\ \ a_{2^{K-1}0}\ \ \dots\ \ a_{2^{K-1}M}\ ]^T$$

and $$b = [b_{10}\ b_{11}\ \ b_{12}\ \ \dots\ b_{1M}\, b_{20}\ \ \dots b_{2M}\ \dots\ \ b_{2^{K-1}0}\ \ \dots\ \ b_{2^{K-1}M}\ ]^T$$

where a and b are vectors of unknown coefficients of dimension $sN \times 1$ and $rN \times 1$.

5.2. Approximation of performance index via LSF and CSF

5.2.1. Approximation of Performance Index via Legendre Scaling Function

We can approximate the performance index of linear time quadratic optimal control problems using Legendre scaling function, substitute equations (5.2) into the objective function. We get

$$J = \frac{1}{2}\int_0^1 a^T\Big(\Phi(t)\otimes I_s)Q(I_s\otimes\Phi^T(t)\Big)a + b^T\Big(\Phi(t)\otimes I_r)R(I_r\otimes\Phi^T(t)\Big)b\,dt \tag{5.4}$$

By applying equation (4.4), we get the equation (5.4) and we can simplify this equation. We get

$$J = \tfrac{1}{2}(a^T(I_N\otimes Q)a + b^T(I_N\otimes R)b) \tag{5.4a}$$

Moreover, equation (5.4a) can be written in quadratic form is as follows

$$J = [a^T \quad b^T]\begin{bmatrix} I_N\otimes Q & O_{Ns\times Nr} \\ O_{Nr\times Ns} & I_N\otimes R\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} \tag{5.4b}$$

5.2.2. Approximation of Performance Index via Chebyshev Scaling Function

We can approximate the performance index of linear time quadratic optimal control problems using Chebyshev scaling function, substitute equations (5.3) into the objective function. We get

$$J = \frac{1}{2}\int_0^1 a^T\Big(\Psi(t)\otimes I_s)Q(I_s\otimes\Psi^T(t)\Big)a + b^T\Big(\Psi(t)\otimes I_r)R(I_r\otimes\Psi^T(t)\Big)b\,dt \tag{5.5}$$

It can be simplify as

$$J = \frac{1}{2}\int_0^1 a^T(\Psi(t)(\Psi^T(t)\otimes Q)a + b^T(\Psi(t)(\Psi^T(t)\otimes R)b\,dt \tag{5.6}$$

By using equation (4.13), we get

$$\text{then} \quad J = \frac{1}{2}(a^T(RR\otimes Q)a + b^T(RR\otimes R)b)$$

It can be write in a quadratic form

$$J = [a^T \quad b^T]\begin{bmatrix} RR\otimes Q & O_{Ns\times Nr} \\ O_{Nr\times Ns} & RR\otimes R \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \tag{5.7}$$

The general form of the linear time invariance quadratic optimal control problems are equation (1.1) with linear constraints equation (1.2). Then the general steps to approximate state, control variable and performance indices using Legendre and Chebyshev scaling function with unknown coefficients of a and b.

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows.

$$a = [a_{10}\ a_{11} \dots a_{1M}\ a_{20}\ a_{21} \dots a_{2M} \dots\ a_{2^{K-1}0} \dots a_{2^{K-1}M}\ ]^T$$

$$b = [b_{10}\ \ b_{11} \dots b_{1M}\ b_{20}\ b_{21} \dots\ b_{2M}\ \dots\ b_{2^{K-1}0} \dots b_{2^{K-1}M}\ ]^T$$

Step (2): Generate the scaling function depending on K and M.

$$\Phi(t) = \left[\Phi_{10}(t)\ \ \Phi_{11}(t) \dots\ \Phi_{1M}(t)\ \ \Phi_{20}(t)\ \ \Phi_{21}(t) \dots\ \Phi_{2M}(t) \dots\ \ \Phi_{2^{K-1}0}(t) \dots\ \ \Phi_{2^{K-1}M}(t)\right]^T$$

Step (3): Approximate the state and control variables.

$$x(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} a_{nm}\Phi_{nm}$$

$$u(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} b_{nm}\Phi_{nm}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{2}}{2^{K/2}} \text{ and } \delta = \frac{\sqrt{\frac{\pi}{2}}}{2^{(K-1)/2}}\ , \qquad \xi_0 = [x_i(0)\ 0\ 0 \dots 0\ |x_i(0)\ 0\ 0 \dots 0]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$\Phi' = \left[\Phi_{10}(t_i) \dots\ \Phi_{1M}(t_i)\ \Phi_{20}(t_i)\ \dots\ \Phi_{2M}(t_i) \dots\ \ \Phi_{2^{K-1}0}(t_i) \dots\ \ \Phi_{2^{K-1}M}(t_i)\right]$$

where $t_i = \frac{i}{2^{K-1}}$, i=1, 2, 3, . . ., $2^{K-1} - 1$

Step (6): Determine the quadratic programming problems or equation (4.9) and (4.19).

Step (7): Solve for the equation (4.9) and (4.19) or quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable.

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value.

Example 5.1 (Abu Haya, 2012)

Find the optimal control $u^*(t)$ and state variable $x^*(t)$ which minimize the performance index.

$$\min \quad J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt$$

$$\text{Subject to } \; x'(t) = -x(t) + u(t), \quad x(0) = 1$$

Solution: To solve example (5.1) by using Legendre scaling function in details and step by step is as follows for K=2. **For M=1**

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a = [a_{10} \; a_{11} \; a_{20} \; a_{21} \;]^T$$

$$b = [b_{10} \; b_{11} \; b_{20} \; b_{21} \;]^T$$

Step (2): Generate the Legendre scaling function depending on K and M.

$$\Phi(t) = [\Phi_{10}(t) \;\; \Phi_{11}(t) \;\; \Phi_{20}(t) \;\; \Phi_{21}(t)]^T$$

Where $\Phi_{10}(t)$, $\Phi_{11}(t)$, $\Phi_{20}(t)$ and $\Phi_{21}(t)$ are in equation (4.2b) and (4.2c)

Step (3): Approximate the state and control variables using unknown coefficients a and b.

$$x(t) = \sum_{n=1}^{2}\sum_{m=0}^{1} a_{nm}\Phi_{nm} = a_{10}\,\Phi_{10} + a_{11}\Phi_{11} + a_{20}\,\Phi_{20} + a_{21}\Phi_{21}$$

$$u(t) = \sum_{n=1}^{2}\sum_{m=0}^{1} b_{nm}\Phi_{nm} = b_{10}\Phi_{10} + b_{11}\Phi_{11} + b_{20}\,\Phi_{20} + b_{21}\Phi_{21}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{2}}{2}, \qquad \xi_0 = [x_1(0)\; 0\; x_1(0)\; 0\;]^T = [1\;0\;1\;0\;]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$t_1 = \frac{1}{2} = 0.5$$

$\Phi' = [\Phi_{10}(0.5) \;\; \Phi_{11}(0.5) - \Phi_{20}(0.5) - \Phi_{21}(0.5)]$ = [1.4142   2.4494 -1.4142 2.4494]

Step (6): Determine the quadratic programming problems or equation (4.9).

Then the equivalent quadratic programming problem is

$$\text{Min } \frac{1}{2}(a_{10}^2 + a_{11}^2 + a_{20}^2 + a_{21}^2 + b_{10}^2 + b_{11}^2 + b_{20}^2 + b_{21}^2)$$

Subject to $\frac{-5}{4}a_{10}+\frac{1}{4\sqrt{3}}a_{11}+\frac{1}{4}b_{10}+\frac{-1}{4\sqrt{3}}b_{11}=\frac{-1}{\sqrt{2}}$

$$\frac{-1}{4\sqrt{3}}a_{10}-a_{11}+\frac{1}{4\sqrt{3}}b_{10}=0$$

$$\frac{-1}{2}a_{10}+\frac{-5}{4}a_{20}+\frac{1}{4\sqrt{3}}a_{21}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}+\frac{-1}{4\sqrt{3}}b_{21}=\frac{-1}{\sqrt{2}}$$

$$\frac{-1}{4\sqrt{3}}a_{20}-a_{21}+\frac{1}{4\sqrt{3}}b_{20}=0$$

$$1.4142a_{10}+2.4494a_{11}-1.4142a_{20}+2.4494a_{21}=0$$

Step (7): Solve the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable in table (1).

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value is below in table (2).

For M=2

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a=[a_{10}\ a_{11}\ a_{12}\ a_{20}\ a_{21}\ a_{22}]^T$$

$$b=[b_{10}\ b_{11}\ b_{12}\ b_{20}\ b_{21}\ b_{22}]^T$$

Step (2): Generate the Legendre scaling function depending on K and M.

$$\Phi(t)=[\Phi_{10}(t)\ \ \Phi_{11}(t)\ \ \Phi_{12}(t)\ \Phi_{20}(t)\ \Phi_{21}(t)\ \Phi_{22}(t)]^T$$

Step (3): Approximate the state and control variables.

$$x(t)=\sum_{n=1}^{2}\sum_{m=0}^{2}a_{nm}\Phi_{nm}=a_{10}\Phi_{10}+a_{11}\Phi_{11}+a_{12}\Phi_{12}+a_{20}\Phi_{20}+a_{21}\Phi_{21}+a_{22}\Phi_{22}$$

$$u(t)=\sum_{n=1}^{2}\sum_{m=0}^{2}b_{nm}\Phi_{nm}=b_{10}\Phi_{10}+b_{11}\Phi_{11}+b_{12}\Phi_{12}+b_{20}\Phi_{20}+b_{21}\Phi_{21}+b_{22}\Phi_{22}$$

Step (4): Find the vector of initial condition

$$\delta=\frac{\sqrt{2}}{2^{K/2}}=\frac{1}{\sqrt{2}},\qquad \xi_0=[1\ 0\ 0\ 1\ 0\ 0]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$t_i=\frac{i}{2^{K-1}}=\frac{1}{2}=0.5$$

$$\Phi'=[\Phi_{10}(0.5)\ \ \Phi_{11}(0.5)\ \ \Phi_{12}(0.5)\ -\Phi_{20}(0.5)\ -\Phi_{21}(0.5)\ -\Phi_{22}(0.5)]$$

$$\Phi' = [1.4142\ \ 2.4494\ \ 3.1622\ -1.4142\ \ 2.4494\ -3.1622]$$

Step (6): Determine the quadratic programming problems or using equation (4.9).

Then the equivalent quadratic programming problem is

$$\text{Min}\ \tfrac{1}{2}(a_{10}^2+a_{11}^2+a_{12}^2+a_{20}^2+a_{21}^2+a_{22}^2+b_{10}^2+b_{11}^2+b_{12}^2+b_{20}^2+b_{21}^2+b_{22}^2)$$

Subject to $\frac{-5}{4}a_{10}+\frac{1}{4\sqrt{3}}a_{11}+\frac{1}{4}b_{10}+\frac{-1}{4\sqrt{3}}b_{11}=\frac{-1}{\sqrt{2}}$

$$\frac{-1}{4\sqrt{3}}a_{10}-a_{11}+\frac{1}{4\sqrt{15}}a_{12}+\frac{1}{4\sqrt{3}}b_{10}+\frac{-1}{4\sqrt{15}}b_{12}=0$$

$$\frac{-1}{4\sqrt{15}}a_{11}+\frac{-1}{4}a_{12}+\frac{1}{4\sqrt{15}}b_{11}=0$$

$$\frac{-1}{2}a_{10}+\frac{-5}{4}a_{20}+\frac{1}{4\sqrt{3}}a_{21}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}+\frac{-1}{4\sqrt{3}}b_{21}=\frac{-1}{\sqrt{2}}$$

$$\frac{-1}{4\sqrt{3}}a_{20}+\frac{1}{4\sqrt{15}}a_{22}+\frac{1}{4\sqrt{3}}b_{20}+\frac{-1}{4\sqrt{15}}b_{22}=0$$

$$\frac{-1}{4\sqrt{15}}a_{21}+\frac{1}{4\sqrt{15}}b_{21}=0$$

$$1.4142a_{10}+2.4494a_{11}+3.1622a_{12}-1.4142a_{20}+2.4494a_{21}-3.1622a_{22}=0$$

Step (7): Solve the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable in table (1).

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value is below in table (2).

For M=3

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a=[a_{10}\ a_{11}\ a_{12}\ a_{13}\ a_{20}\ a_{21}\ a_{22}\ a_{23}]^T$$

$$b=[b_{10}\ b_{11}\ b_{12}\ b_{13}\ b_{20}\ b_{21}\ b_{22}\ b_{23}]^T$$

Step (2): Generate the Legendre scaling function depending on K and M.

$$\Phi(t)=[\Phi_{10}(t)\ \ \Phi_{11}(t)\ \ \Phi_{12}(t)\ \ \Phi_{13}(t)\ \ \Phi_{20}(t)\ \Phi_{21}(t)\ \Phi_{22}(t)\Phi_{23}(t)]^T$$

Step (3): Approximate the state and control variables.

$$x(t)=a_{10}\Phi_{10}+a_{11}\Phi_{11}+a_{12}\Phi_{12}+a_{13}\Phi_{13}+a_{20}\Phi_{20}+a_{21}\Phi_{21}+a_{22}\Phi_{22}+a_{23}\Phi_{23}$$

$$u(t)=b_{10}\Phi_{10}+b_{11}\Phi_{11}+b_{12}\Phi_{12}+b_{13}\Phi_{13}+b_{20}\Phi_{20}+b_{21}\Phi_{21}+b_{22}\Phi_{22}+b_{23}\Phi_{23}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{2}}{2^{K/2}} = \frac{1}{\sqrt{2}}, \qquad \xi_0 = [1\ 0\ 0\ 0\ 1\ \ 0\ 0\ 0]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$t_i = \frac{i}{2^{K-1}} = \frac{1}{2}$$

$$\Phi' = [\Phi_{10}(0.5)\ \ \Phi_{11}(0.5)\ \Phi_{12}(0.5)\ \Phi_{13}(0.5)\ -\Phi_{20}(0.5)\ -\Phi_{21}(0.5)\ -\Phi_{22}(0.5) - \Phi_{23}(0.5)]$$

$$\Phi' = [1.4142\ \ 2.4494\ \ 3.1622\ \ 3.7416 - 1.4142\ \ 2.4494\ \ -3.1622\ \ 3.7416]$$

Step (6): Determine the quadratic programming problems or equation (4.9).

Then the equivalent quadratic programming problem is

Min $\quad \frac{1}{2}(a_{10}^2 + a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{20}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + b_{10}^2 + b_{11}^2 + b_{12}^2 + b_{13}^2 + b_{20}^2 + b_{21}^2 + b_{22}^2 + b_{23}^2)$

Subject to $\quad \frac{-5}{4}a_{10} + \frac{1}{4\sqrt{3}}a_{11} + \frac{1}{4}b_{10} - \frac{1}{4\sqrt{3}}b_{11} = \frac{-1}{\sqrt{2}}$

$$\frac{-1}{4\sqrt{3}}a_{10} - a_{11} + \frac{1}{4\sqrt{15}}a_{12} + \frac{1}{4\sqrt{3}}b_{10} - \frac{1}{4\sqrt{15}}b_{12} = 0$$

$$\frac{-1}{4\sqrt{15}}a_{11} - a_{12} + \frac{1}{4\sqrt{35}}a_{13} + \frac{1}{4\sqrt{15}}b_{11} - \frac{1}{4\sqrt{35}}b_{13} = 0$$

$$\frac{-1}{4\sqrt{35}}a_{12} - a_{13} + \frac{1}{4\sqrt{35}}b_{12} = 0$$

$$\frac{-1}{2}a_{10} - \frac{5}{4}a_{20} + \frac{1}{4\sqrt{3}}a_{21} + \frac{1}{2}b_{10} + \frac{1}{4}b_{20} - \frac{1}{4\sqrt{3}}b_{21} = \frac{-1}{\sqrt{2}}$$

$$\frac{-1}{4\sqrt{3}}a_{20} - a_{21} + \frac{1}{4\sqrt{15}}a_{22} + \frac{1}{4\sqrt{3}}b_{20} - \frac{1}{4\sqrt{15}}b_{22} = 0$$

$$\frac{-1}{4\sqrt{15}}a_{21} - a_{22} + \frac{1}{4\sqrt{35}}a_{23} + \frac{1}{4\sqrt{15}}b_{21} - \frac{1}{4\sqrt{35}}b_{23} = 0$$

$$-\frac{1}{4\sqrt{35}}a_{22} - a_{23} + \frac{1}{4\sqrt{35}}b_{22} = 0$$

$$1.4142a_{10} + 2.4494a_{11} + 3.1622a_{12} + 3.7416\,a_{13} - 1.4142a_{20} + 2.4494a_{21} - 3.1622a_{22} + 3.7416a_{23} = 0$$

Step (7): Solve the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable in table (1).

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value is below in table (2).

To solve example (5.1) by using Chebyshev scaling function in details and step by step is as follows for K=2. **For M=1**

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a = [a_{10}\ a_{11}\ a_{20}\ a_{21}\ ]^T$$

$$b = [b_{10}\ \ b_{11}\ b_{20}\ b_{21}\ ]^T$$

Step (2): Generate the Chebyshev scaling function depending on K and M.

$$\Psi(t) = [\Psi_{10}(t)\ \ \Psi_{11}(t)\ \ \Psi_{20}(t)\ \ \Psi_{21}(t)\ ]^T$$

where $\Psi_{10}(t)$, $\Psi_{11}(t)$, $\Psi_{20}(t)$ and $\Psi_{21}(t)$ are in equation (4.11b) and (4.11c)

Step (3): Approximate the state and control variables using unknown coefficients a and b.

$$x(t) = \sum_{n=1}^{2}\sum_{m=0}^{1} a_{nm}\Psi_{nm} = a_{10}\ \Psi_{10} + a_{11}\Psi_{11}\ + a_{20}\ \Psi_{20} + a_{21}\Psi_{21}$$

$$u(t) = \sum_{n=1}^{2}\sum_{m=0}^{1} b_{nm}\Psi_{nm} = b_{10}\Psi_{10} + b_{11}\Psi_{11} + b_{20}\ \Psi_{20} + b_{21}\Psi_{21}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{\pi}}{2}, \qquad g_0 = [x_1(0)\ 0\ x_1(0)\ 0\ ]^T = [1\ 0\ 1\ 0\ ]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$t_1 = \frac{1}{2} = 0.5$$

$\Psi' = [\Psi_{10}(0.5)\ \ \Psi_{11}(0.5) - \Psi_{20}(0.5) - \Psi_{21}(0.5)]$ = [1.1284 1.5958 -1.1284 1.5958]

Step (6): Determine the quadratic programming problems or equation (4.19).

Then the equivalent quadratic programming problem is

$$\text{Min } \frac{1}{2\pi}(2a_{10}^2 + \frac{4}{3}a_{11}^2 + 2a_{20}^2 + \frac{4}{3}a_{21}^2 + 2b_{10}^2 + \frac{4}{3}b_{11}^2 + 2b_{20}^2 + \frac{4}{3}b_{21}^2)$$

Subject to
$$\frac{-5}{4}a_{10} + \frac{1}{8\sqrt{2}}a_{11} + \frac{1}{4}b_{10} + \frac{-1}{8\sqrt{2}}b_{11} = \frac{-\sqrt{\pi}}{2}$$

$$\frac{-1}{4\sqrt{2}}a_{10} - a_{11} + \frac{1}{4\sqrt{2}}b_{10} = 0$$

$$\frac{-1}{2}a_{10} - \frac{5}{4}a_{20} + \frac{1}{8\sqrt{2}}a_{21} + \frac{1}{2}b_{10} + \frac{1}{4}b_{20} - \frac{1}{8\sqrt{2}}b_{21} = \frac{-\sqrt{\pi}}{2}$$

$$\frac{-1}{8\sqrt{2}}a_{20} - a_{21} + \frac{1}{8\sqrt{2}}b_{20} = 0$$

$$1.1284a_{10} + 1.5958a_{11} - 1.1284a_{20} + 1.5958a_{21} = 0$$

Step (7): Solve the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable in table (1).

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value is below in table (2).

For M=2

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a = [a_{10}\ a_{11}\ \ a_{12}\ a_{20}\ a_{21}\ \ a_{22}]^T$$

$$b = [b_{10}\ \ b_{11}\ \ b_{12}\ \ b_{20}\ b_{21}\ \ b_{22}]^T$$

Step (2): Generate the Chebyshev scaling function depending on K and M.

$$\Psi(t) = [\Psi_{10}(t)\ \ \Psi_{11}(t)\ \ \ \Psi_{12}(t)\ \ \Psi_{20}(t)\ \ \Psi_{21}(t)\ \ \ \Psi_{22}(t)]^T$$

where $\Psi_{10}(t)$, $\Psi_{11}(t), \Psi_{12}(t), \Psi_{20}(t)$, $\Psi_{21}(t)$ and $\Psi_{22}(t)$ are in equation (4.11b) and (4.11c)

Step (3): Approximate the state and control variables using unknown coefficients a and b.

$$x(t) = \sum_{n=1}^{2}\sum_{m=0}^{2} a_{nm}\Psi_{nm} = a_{10}\ \Psi_{10} + a_{11}\Psi_{11} + a_{12}\Psi_{12}\ + a_{20}\ \Psi_{20} + a_{21}\Psi_{21} + a_{22}\Psi_{22}$$

$$u(t) = \sum_{n=1}^{2}\sum_{m=0}^{2} b_{nm}\Psi_{nm} = b_{10}\Psi_{10} + b_{11}\Psi_{11} + b_{12}\Psi_{12} + b_{20}\ \Psi_{20} + b_{21}\Psi_{21} + b_{22}\Psi_{22}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{\pi}}{2}, \qquad g_0 = [x_1(0)\ 0\ x_1(0)\ 0\ ]^T = [1\ 0\ 0\ 1\ 0\ 0\ ]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$t_1 = \frac{1}{2} = 0.5$$

$$\Psi' = [\Psi_{10}(0.5)\ \ \Psi_{11}(0.5)\ \ \Psi_{12}(0.5)\ \ - \Psi_{20}(0.5)\ - \Psi_{21}(0.5)\ - \Psi_{22}(0.5)]$$

$$\Psi' = [1.1284\ \ 1.5958\ \ 1.5958\ \ -1.1284\ \ 1.5958\ -1.5958\ \ ]$$

Step (6): Determine the quadratic programming problems or equation (4.19).

Then the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2\pi}\Big(2a_{10}^2-\frac{4\sqrt{2}}{3}a_{10}a_{12}+\frac{4}{3}a_{11}^2+0.594\pi a_{12}^2+2a_{20}^2-\frac{4\sqrt{2}}{3}a_{20}a_{22}+\frac{4}{3}a_{21}^2+0.594\pi a_{22}^2$$

$$+2b_{10}^2-\frac{4\sqrt{2}}{3}b_{10}b_{12}+\frac{4}{3}b_{11}^2+0.594\pi b_{12}^2+2b_{20}^2-\frac{4\sqrt{2}}{3}b_{20}b_{22}+\frac{4}{3}b_{21}^2+0.594\pi b_{22}^2\Big)$$

Subject to $\frac{-5}{4}a_{10}+\frac{1}{8\sqrt{2}}a_{11}+\frac{1}{6\sqrt{2}}a_{12}+\frac{1}{4}b_{10}-\frac{1}{8\sqrt{2}}b_{11}-\frac{1}{6\sqrt{2}}b_{12}=\frac{-\sqrt{\pi}}{2}$

$$\frac{-1}{4\sqrt{2}}a_{10}-a_{11}+\frac{1}{8}a_{12}+\frac{1}{4\sqrt{2}}b_{10}-\frac{1}{8}b_{12}=0$$

$$\frac{-1}{16}a_{11}-a_{12}+\frac{1}{16}b_{11}=0$$

$$\frac{-1}{2}a_{10}+\frac{\sqrt{2}}{6}a_{12}-\frac{5}{4}a_{20}+\frac{1}{8\sqrt{2}}a_{21}+\frac{1}{6\sqrt{2}}a_{22}+\frac{1}{2}b_{10}-\frac{\sqrt{2}}{6}b_{12}+\frac{1}{4}b_{20}-\frac{1}{8\sqrt{2}}b_{21}-\frac{1}{6\sqrt{2}}b_{21}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{-1}{4\sqrt{2}}a_{20}-a_{21}+\frac{1}{8}a_{22}+\frac{1}{4\sqrt{2}}b_{20}-\frac{1}{8}b_{22}=0$$

$$\frac{-1}{16}a_{21}-a_{22}+\frac{1}{16}b_{21}=0$$

$$1.1284a_{10}+1.5958a_{11}+1.5958a_{12}-1.1284a_{20}+1.5958a_{21}-1.5958a_{22}=0$$

Step (7): Solve the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable in table (1).

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value is below in table (2).

For M=3

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a=[a_{10}\ a_{11}\ a_{12}\ a_{13}\ a_{20}\ a_{21}\ a_{22}\ a_{23}]^T$$

$$b=[b_{10}\ b_{11}\ b_{12}\ b_{13}\ b_{20}\ b_{21}\ b_{22}\ b_{23}]^T$$

Step (2): Generate the Chebyshev scaling function depending on K and M.

$$\Psi(t)=[\Psi_{10}(t)\ \Psi_{11}(t)\ \Psi_{12}(t)\ \Psi_{13}(t)\ \Psi_{20}(t)\ \Psi_{21}(t)\ \Psi_{22}(t)\ \Psi_{23}(t)]^T$$

where $\Psi_{10}(t)$, $\Psi_{11}(t)$, $\Psi_{12}(t)$, $\Psi_{13}(t)$, $\Psi_{20}(t)$, $\Psi_{21}(t)$, $\Psi_{22}(t)$ and $\Psi_{23}(t)$ are in equation (4.11b) and (4.11c)

Step (3): Approximate the state and control variables using unknown coefficients a and b.

$$x(t)=a_{10}\Psi_{10}+a_{11}\Psi_{11}+a_{12}\Psi_{12}+a_{13}\Psi_{13}+a_{20}\Psi_{20}+a_{21}\Psi_{21}+a_{22}\Psi_{22}+a_{23}\Psi_{23}$$

$$u(t)=b_{10}\Psi_{10}+b_{11}\Psi_{11}+b_{12}\Psi_{12}+b_{13}\Psi_{13}+b_{20}\Psi_{20}+b_{21}\Psi_{21}+b_{22}\Psi_{22}+b_{23}\Psi_{23}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{\pi}}{2}, \qquad g_0 = [x_1(0)\ 0\ \ x_1(0)\ 0\ ]^T = [1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ ]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$t_1 = \frac{1}{2} = 0.5$$

$$\Psi' = [\Psi_{10}(0.5)\ \ \Psi_{11}(0.5)\ \ \Psi_{12}(0.5)\ \ \Psi_{13}(0.5)\ -\Psi_{20}(0.5)\ -\Psi_{21}(0.5)\ -\Psi_{22}(0.5)\ -\Psi_{23}(0.5)]$$

Ψ'= [1.1284 1.5958 1.5958 1.5958   -1.1284 1.5958 -1.5958 -1.5958]

Step (6): Determine the quadratic programming problems or equation (4.19).

Then the equivalent quadratic programming problem is

Min $\frac{1}{2\pi}(2a_{10}^2 - \frac{4\sqrt{2}}{3}a_{10}a_{12} + \frac{4}{3}a_{11}^2 + \frac{28}{15}a_{12}^2 + \frac{8}{5}a_{11}a_{13} + \frac{68}{35}a_{13}^2 + 2a_{20}^2 - \frac{4\sqrt{2}}{3}a_{20}a_{22} + \frac{4}{3}a_{21}^2 + \frac{28}{15}a_{22}^2 + \frac{8}{5}a_{21}a_{23} + \frac{68}{35}a_{23}^2 + 2b_{10}^2 - \frac{4\sqrt{2}}{3}b_{10}b_{12} + \frac{4}{3}b_{11}^2 + \frac{28}{15}b_{12}^2 + \frac{8}{5}b_{11}b_{13} + \frac{68}{35}b_{13}^2 + 2b_{20}^2 - \frac{4\sqrt{2}}{3}b_{20}b_{22} + \frac{4}{3}b_{21}^2 + \frac{28}{15}b_{22}^2 + \frac{8}{5}b_{21}b_{23} + \frac{68}{35}b_{23}^2$

Subject to $\frac{-5}{4}a_{10} + \frac{1}{8\sqrt{2}}a_{11} + \frac{1}{6\sqrt{2}}a_{12} + \frac{1}{32\sqrt{2}}a_{13} + \frac{1}{4}b_{10} - \frac{1}{8\sqrt{2}}b_{11} - \frac{1}{6\sqrt{2}}b_{12} + \frac{1}{32\sqrt{2}}b_{13} = \frac{-\sqrt{\pi}}{2}$

$$\frac{-1}{4\sqrt{2}}a_{10} - a_{11} + \frac{1}{8}a_{12} + \frac{1}{4\sqrt{2}}b_{10} - \frac{1}{8}b_{12} = 0$$

$$\frac{-1}{16}a_{11} - a_{12} + \frac{1}{16}b_{11} = 0$$

$$\frac{-1}{24}a_{12} - a_{13} + \frac{1}{24}b_{12} = 0$$

$$\frac{-1}{2}a_{10} + \frac{\sqrt{2}}{6}a_{12} - \frac{5}{4}a_{20} + \frac{1}{8\sqrt{2}}a_{21} + \frac{1}{6\sqrt{2}}a_{22} + \frac{1}{32\sqrt{2}}a_{23} + \frac{1}{2}b_{10} - \frac{\sqrt{2}}{6}b_{12} + \frac{1}{4}b_{20} - \frac{1}{8\sqrt{2}}b_{21} - \frac{1}{6\sqrt{2}}b_{21} - \frac{1}{32\sqrt{2}}a_{23} = \frac{-\sqrt{\pi}}{2}$$

$$\frac{-1}{4\sqrt{2}}a_{20} - a_{21} + \frac{1}{8}a_{22} + \frac{1}{4\sqrt{2}}b_{20} - \frac{1}{8}b_{22} = 0$$

$$\frac{-1}{16}a_{21} - a_{22} + \frac{1}{16}b_{21} = 0$$

$$\frac{-1}{24}a_{22} - a_{23} + \frac{1}{24}b_{22} = 0$$

$$1.1284a_{10} + 1.5958a_{11} + 1.5958a_{12} + 1.5958a_{12} - 1.1284a_{20} + 1.5958a_{21} - 1.5958a_{22} - 1.5958a_{23} = 0$$

Step (7): Solve the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable in table (1).

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value is below in table (2).

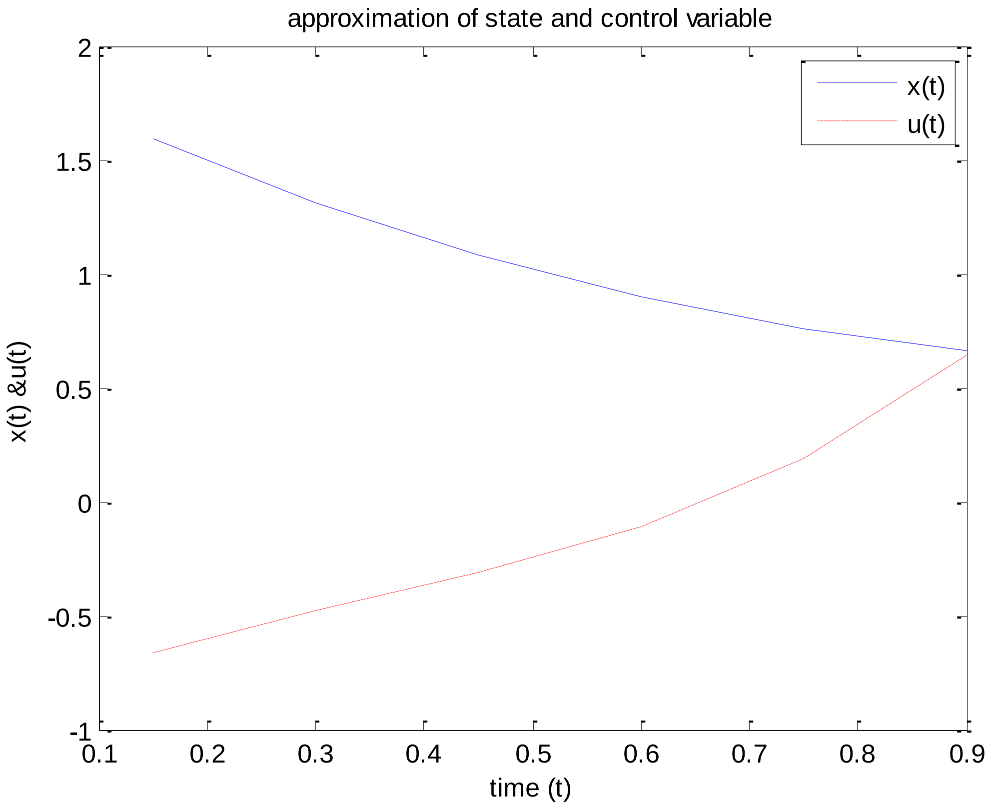

Figure 4. The approximation of state and control variable using Legendre scaling function

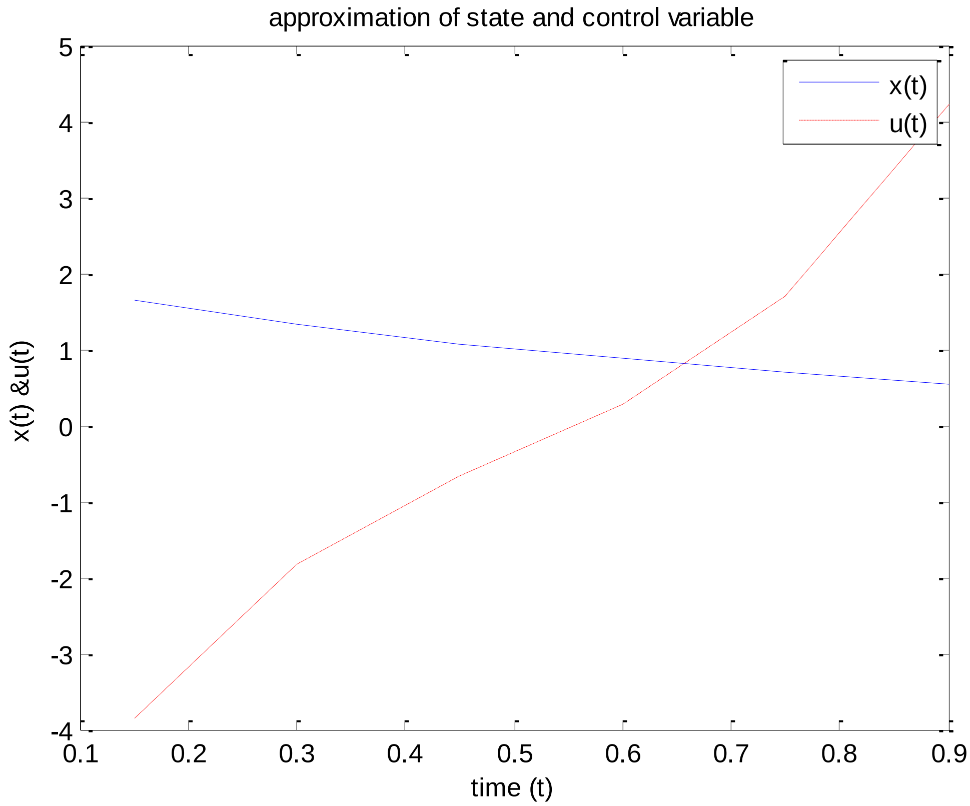

Figure 5. The approximation of state and control variable using Chebyshev scaling function

The approximation value of state variable and control variable in the linear time invariant system, were presented in Figure 4 and Figure 5 using Legendre scaling function and Chebyshev scaling function respectively. We have considered parameter values K=2, with polynomial of degree three. We observed that as the time interval, $t \in (0, 1)$ the value of state variable decreases and control variable increases. MATLAB is used to plot the graphs to show the best result of problems. The graphs represent the best approximation at each system and the best value of state and control variable by the run considering numerical results of the solutions are shown in Figure 4 and Figure 5.

Table 1. A comparison between Legendre and Chebyshev scaling function with the approximations of performance index of example 5.1.

Optimal value (Zeid and Yousefi, 2016)	0.1929092981			
Order of poly	Performance index (J)			
	Legendre	Error	Chebyshev	Error
M=1	0.194179279497372	1.27×10^{-3}	0.196333244960920	3.42×10^{-3}
M=2	0.192998674286855	8.94×10^{-5}	0.193000989090520	9.17×10^{-5}
M=3	0.192909322011399	2.39×10^{-8}	0.192934130143056	2.48×10^{-5}

Example (5.1) solved by Legendre scaling function and Chebyshev scaling function. As M=1 to M=3 the approximation value is approaching to the optimal value obtained by Zeid and Yousefi (2016). From this table we conclude that Legendre scaling function is better than Chebyshev scaling function.

Example 5.2 (Kafash and Delavarkhalafi, 2015)

Find the optimal control $u^*(t)$ and state variable $x^*(t)$ which minimize the performance index.

$$\min \quad J = \int_0^1 (x^2(t) + \frac{1}{2}u^2(t))dt$$

$$\text{Subject to} \qquad x'(t) = \frac{1}{2}x(t) + u(t), \quad x(0) = 1$$

Solution:

This optimal control problem transform into quadratic programming problem is similar steps to the above example 5.1 with the same scaling parameter (K=2).

Using Legendre scaling function for M=1

The equivalent quadratic programming problem is

$$\text{Min } \frac{1}{2}(2a_{10}^2 + 2a_{11}^2 + 2a_{20}^2 + 2a_{21}^2 + b_{10}^2 + b_{11}^2 + b_{20}^2 + b_{21}^2)$$

$$\text{Subject to} \qquad \frac{-7}{8}a_{10} - \frac{1}{8\sqrt{3}}a_{11} + \frac{1}{4}b_{10} + \frac{-1}{4\sqrt{3}}b_{11} = \frac{-1}{\sqrt{2}}$$

$$\frac{1}{8\sqrt{3}}a_{10} - a_{11} + \frac{1}{4\sqrt{3}}b_{10} = 0$$

$$\frac{1}{4}a_{10} - \frac{7}{8}a_{20} - \frac{1}{8\sqrt{3}}a_{21} + \frac{1}{2}b_{10} + \frac{1}{4}b_{20} + \frac{-1}{4\sqrt{3}}b_{21} = \frac{-1}{\sqrt{2}}$$

$$\frac{1}{8\sqrt{3}}a_{20}-a_{21}+\frac{1}{4\sqrt{3}}b_{20}=0$$

$$1.4142a_{10}+2.4494a_{11}-1.4142a_{20}+2.4494a_{21}=0$$

For M=2 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2}(2a_{10}^2+2a_{11}^2+2a_{12}^2+2a_{20}^2+2a_{21}^2+2a_{22}^2+b_{10}^2+b_{11}^2+b_{12}^2+b_{20}^2+b_{21}^2+b_{22}^2)$$

Subject to $\frac{-7}{8}a_{10}-\frac{1}{8\sqrt{3}}a_{11}+\frac{1}{4}b_{10}+\frac{-1}{4\sqrt{3}}b_{11}=\frac{-1}{\sqrt{2}}$

$$\frac{1}{8\sqrt{3}}a_{10}-a_{11}+\frac{1}{8\sqrt{15}}a_{12}+\frac{1}{4\sqrt{3}}b_{10}+\frac{-1}{4\sqrt{15}}b_{12}=0$$

$$\frac{1}{8\sqrt{15}}a_{11}-a_{12}+\frac{1}{4\sqrt{15}}b_{11}=0$$

$$\frac{1}{4}a_{10}+\frac{-7}{8}a_{20}-\frac{1}{8\sqrt{3}}a_{21}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}+\frac{-1}{4\sqrt{3}}b_{21}=\frac{-1}{\sqrt{2}}$$

$$\frac{1}{8\sqrt{3}}a_{20}-a_{21}-\frac{1}{8\sqrt{15}}a_{22}+\frac{1}{4\sqrt{3}}b_{20}+\frac{-1}{4\sqrt{15}}b_{22}=0$$

$$\frac{1}{8\sqrt{15}}a_{21}+\frac{1}{4\sqrt{15}}b_{21}=0$$

$$1.4142a_{10}+2.4494a_{11}+3.1622a_{12}-1.4142a_{20}+2.4494a_{21}-3.1622a_{22}=0$$

For M=3 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2}(2a_{10}^2+2a_{11}^2+2a_{12}^2+2a_{13}^2+2a_{20}^2+2a_{21}^2+2a_{22}^2+2a_{23}^2+b_{10}^2+b_{11}^2+b_{12}^2+b_{13}^2+b_{20}^2+b_{21}^2+b_{22}^2+b_{23}^2)$$

Subject to $\frac{-7}{8}a_{10}-\frac{1}{8\sqrt{3}}a_{11}+\frac{1}{4}b_{10}-\frac{1}{4\sqrt{3}}b_{11}=\frac{-1}{\sqrt{2}}$

$$\frac{1}{8\sqrt{3}}a_{10}-a_{11}-\frac{1}{8\sqrt{15}}a_{12}+\frac{1}{4\sqrt{3}}b_{10}-\frac{1}{4\sqrt{15}}b_{12}=0$$

$$\frac{1}{8\sqrt{15}}a_{11}-a_{12}-\frac{1}{8\sqrt{35}}a_{13}+\frac{1}{4\sqrt{15}}b_{11}-\frac{1}{4\sqrt{35}}b_{13}=0$$

$$\frac{1}{8\sqrt{35}}a_{12}-a_{13}+\frac{1}{4\sqrt{35}}b_{12}=0$$

$$\frac{1}{4}a_{10}-\frac{7}{8}a_{20}-\frac{1}{8\sqrt{3}}a_{21}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}-\frac{1}{4\sqrt{3}}b_{21}=\frac{-1}{\sqrt{2}}$$

$$\frac{1}{8\sqrt{3}}a_{20}-a_{21}-\frac{1}{8\sqrt{15}}a_{22}+\frac{1}{4\sqrt{3}}b_{20}-\frac{1}{4\sqrt{15}}b_{22}=0$$

$$\frac{1}{8\sqrt{15}}a_{21}-a_{22}-\frac{1}{8\sqrt{35}}a_{23}+\frac{1}{4\sqrt{15}}b_{21}-\frac{1}{4\sqrt{35}}b_{23}=0$$

$$\frac{1}{8\sqrt{35}}a_{22} - a_{23} + \frac{1}{4\sqrt{35}}b_{22} = 0$$

$$1.4142a_{10} + 2.4494a_{11} + 3.1622a_{12} + 3.7416\,a_{13} - 1.4142a_{20} + 2.4494a_{21} - 3.1622a_{22} + 3.7416a_{23} = 0$$

Using Chebyshev scaling function

For M=1 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2\pi}(4a_{10}^2 + \frac{8}{3}a_{11}^2 + 4a_{20}^2 + \frac{8}{3}a_{21}^2 + 2b_{10}^2 + \frac{4}{3}b_{11}^2 + 2b_{20}^2 + \frac{4}{3}b_{21}^2)$$

Subject to $\frac{-7}{8}a_{10} - \frac{1}{16\sqrt{2}}a_{11} + \frac{1}{4}b_{10} + \frac{-1}{8\sqrt{2}}b_{11} = \frac{-\sqrt{\pi}}{2}$

$$\frac{1}{8\sqrt{2}}a_{10} - a_{11} + \frac{1}{4\sqrt{2}}b_{10} = 0$$

$$\frac{1}{4}a_{10} - \frac{7}{8}a_{20} - \frac{1}{16\sqrt{2}}a_{21} + \frac{1}{2}b_{10} + \frac{1}{4}b_{20} - \frac{1}{8\sqrt{2}}b_{21} = \frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{8\sqrt{2}}a_{20} - a_{21} + \frac{1}{8\sqrt{2}}b_{20} = 0$$

$$1.1284a_{10} + 1.5958a_{11} - 1.1284a_{20} + 1.5958a_{21} = 0$$

For M=2 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2\pi}(4a_{10}^2 - \frac{8\sqrt{2}}{3}a_{10}a_{12} + \frac{8}{3}a_{11}^2 + 1.188\pi a_{12}^2 + 4a_{20}^2 - \frac{8\sqrt{2}}{3}a_{20}a_{22} + \frac{8}{3}a_{21}^2 + 1.188\pi a_{22}^2$$

$$+2b_{10}^2 - \frac{4\sqrt{2}}{3}b_{10}b_{12} + \frac{4}{3}b_{11}^2 + 0.594\pi b_{12}^2 + 2b_{20}^2 - \frac{4\sqrt{2}}{3}b_{20}b_{22} + \frac{4}{3}b_{21}^2 + 0.594\pi b_{22}^2)$$

Subject to $\frac{-7}{8}a_{10} - \frac{1}{8\sqrt{2}}a_{11} - \frac{1}{12\sqrt{2}}a_{12} + \frac{1}{4}b_{10} - \frac{1}{8\sqrt{2}}b_{11} - \frac{1}{6\sqrt{2}}b_{12} = \frac{-\sqrt{\pi}}{2}$

$$\frac{1}{8\sqrt{2}}a_{10} - a_{11} - \frac{1}{16}a_{12} + \frac{1}{4\sqrt{2}}b_{10} - \frac{1}{8}b_{12} = 0$$

$$\frac{1}{32}a_{11} - a_{12} + \frac{1}{16}b_{11} = 0$$

$$\frac{1}{4}a_{10} - \frac{\sqrt{2}}{12}a_{12} - \frac{7}{8}a_{20} - \frac{1}{16\sqrt{2}}a_{21} - \frac{1}{12\sqrt{2}}a_{22} + \frac{1}{2}b_{10} - \frac{\sqrt{2}}{6}b_{12} + \frac{1}{4}b_{20} - \frac{1}{8\sqrt{2}}b_{21} - \frac{1}{6\sqrt{2}}b_{21} = \frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{8\sqrt{2}}a_{20} - a_{21} - \frac{1}{16}a_{22} + \frac{1}{4\sqrt{2}}b_{20} - \frac{1}{8}b_{22} = 0$$

$$\frac{1}{32}a_{21} - a_{22} + \frac{1}{16}b_{21} = 0$$

$$1.1284a_{10} + 1.5958a_{11} + 1.5958a_{12} - 1.1284a_{20} + 1.5958a_{21} - 1.5958a_{22} = 0$$

For M=3 the equivalent quadratic programming problem is

Min $\frac{1}{2\pi}(4a_{10}^2 - \frac{8\sqrt{2}}{3}a_{10}a_{12} + \frac{8}{3}a_{11}^2 + \frac{56}{15}a_{12}^2 + \frac{16}{5}a_{11}a_{13} + \frac{136}{35}a_{13}^2 + 4a_{20}^2 - \frac{8\sqrt{2}}{3}a_{20}a_{22} + \frac{8}{3}a_{21}^2 + \frac{56}{15}a_{22}^2 + \frac{16}{5}a_{21}a_{23} + \frac{136}{35}a_{23}^2 + 2b_{10}^2 - \frac{4\sqrt{2}}{3}b_{10}b_{12} + \frac{4}{3}b_{11}^2 + \frac{28}{15}b_{12}^2 + \frac{8}{5}b_{11}b_{13} + \frac{68}{35}b_{13}^2 + 2b_{20}^2 - \frac{4\sqrt{2}}{3}b_{20}b_{22} + \frac{4}{3}b_{21}^2 + \frac{28}{15}b_{22}^2 + \frac{8}{5}b_{21}b_{23} + \frac{68}{35}b_{23}^2$

Subject to $\frac{5}{8}a_{10} - \frac{1}{16\sqrt{2}}a_{11} - \frac{1}{12\sqrt{2}}a_{12} - \frac{1}{64\sqrt{2}}a_{13} + \frac{1}{4}b_{10} - \frac{1}{8\sqrt{2}}b_{11} - \frac{1}{6\sqrt{2}}b_{12} + \frac{1}{32\sqrt{2}}b_{13} = \frac{-\sqrt{\pi}}{2}$

$$\frac{1}{8\sqrt{2}}a_{10} - a_{11} - \frac{1}{16}a_{12} + \frac{1}{4\sqrt{2}}b_{10} - \frac{1}{8}b_{12} = 0$$

$$\frac{1}{32}a_{11} - a_{12} + \frac{1}{16}b_{11} = 0$$

$$\frac{1}{48}a_{12} - a_{13} + \frac{1}{24}b_{12} = 0$$

$$\frac{1}{4}a_{10} - \frac{\sqrt{2}}{12}a_{12} - \frac{3}{8}a_{20} - \frac{1}{16\sqrt{2}}a_{21} - \frac{1}{12\sqrt{2}}a_{22} - \frac{1}{64\sqrt{2}}a_{23} + \frac{1}{2}b_{10} - \frac{\sqrt{2}}{6}b_{12} + \frac{1}{4}b_{20} - \frac{1}{8\sqrt{2}}b_{21} - \frac{1}{6\sqrt{2}}b_{21} - \frac{1}{32\sqrt{2}}a_{23} = \frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{8\sqrt{2}}a_{20} - a_{21} - \frac{1}{16}a_{22} + \frac{1}{4\sqrt{2}}b_{20} - \frac{1}{8}b_{22} = 0$$

$$\frac{1}{32}a_{21} - a_{22} + \frac{1}{16}b_{21} = 0$$

$$\frac{1}{48}a_{22} - a_{23} + \frac{1}{24}b_{22} = 0$$

$$1.1284a_{10} + 1.5958a_{11} + 1.5958a_{12} + 1.5958a_{12} - 1.1284a_{20} + 1.5958a_{21} - 1.5958a_{22} - 1.5958a_{23} = 0$$

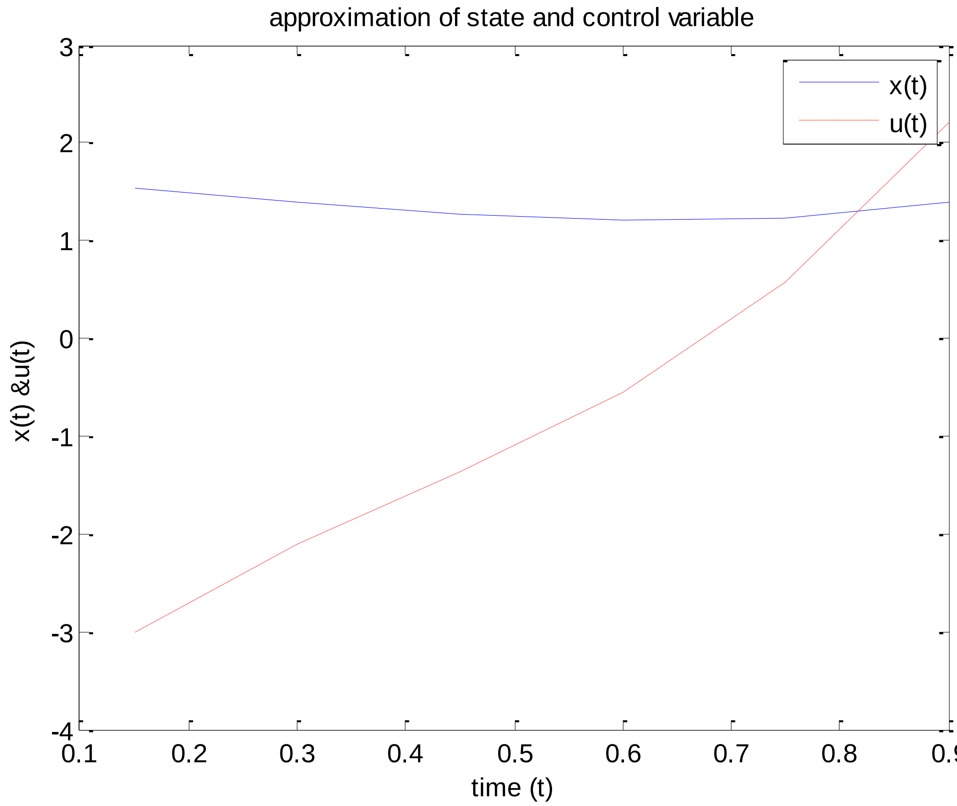

Figure 6. The approximation of state and control variable using Legendre scaling function

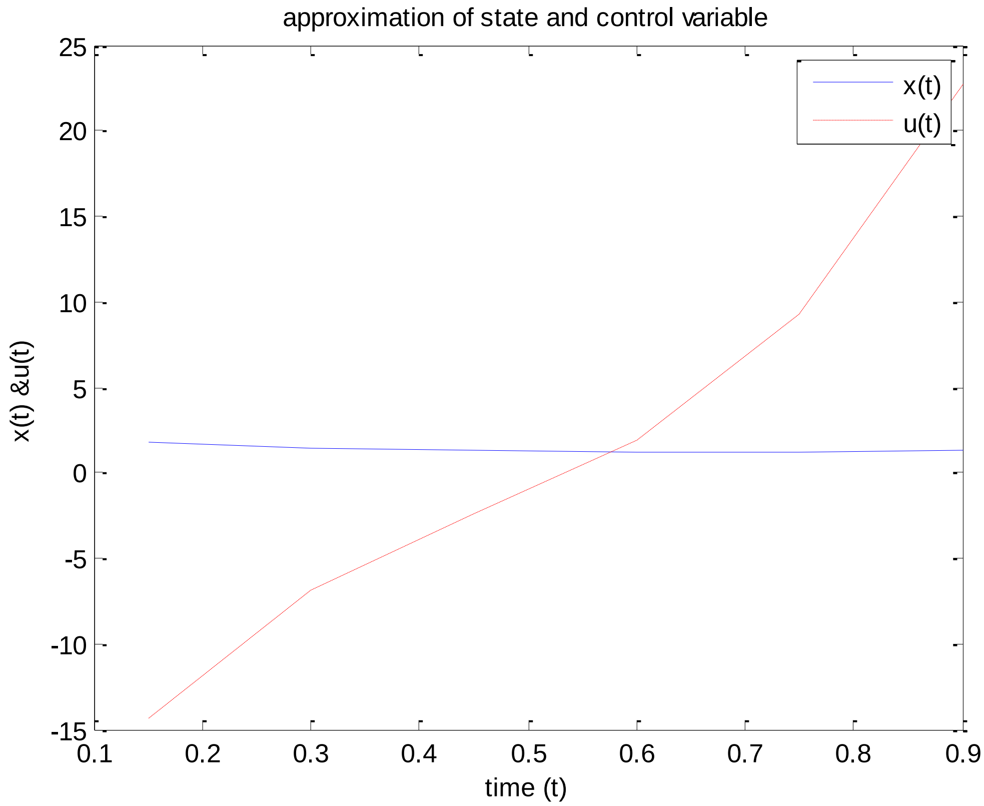

Figure 7. The approximation of state and control variable using Chebyshev scaling function

The approximations of state variables and control variables with polynomial of degree three are shown in Figures 6 and Figure 7 using Legendre scaling function and Chebyshev scaling function respectively. Since the system is linear time invariant; so the control variables increases with the time interval, $t \in (0, 1)$. But the value of state variables almost constant because the coefficient of the sate variable is half of the normal state variable.

Table 2. A comparison between Legendre and Chebyshev scaling function with the approximations of performance index of example 5.2.

Optimal value (Kafash and Delavarkhalafi, 2015)		0.8641644978		
Order of poly	Performance index (J)			
	Legendre	Error	Chebyshev	Error
M=1	0.864473960577410	3.09×10^{-4}	0.859873111505597	4.29×10^{-3}
M=2	0.864198662433090	3.42×10^{-5}	0.864217766347930	5.33×10^{-5}
M=3	0.864164508655142	1.09×10^{-8}	0.864123208655142	4.12×10^{-5}

The results obtained for the linear time invariance optimal control systems with quadratic performance index shows that the Legendre and Chebyshev scaling functions works for finding the approximation values of performance index. Table (2) shows that, for the polynomial of scaling function with degree three and the time interval $t \in [0,1]$, the absolute error is less (almost zero) with the Legendre scaling functions than with the Chebyshev scaling functions method, which yields a small error compared with the optimal solutions of the problem. Hence, the Legendre scaling function method is more suitable for solving the linear time invariance optimal control systems.

5.3. Linear Time Varying Quadratic Optimal Control Problems

5.3.1. Parameterization $\mathrm{A}(\mathbf{t})$ and $\mathrm{B}(\mathbf{t})$ of Legendre Scaling Function

The parameterization of time varying matrices $\mathbf{A}(\mathbf{t})$ and $\mathbf{B}(\mathbf{t})$ from the problem of equation (1.4) can be expressed in terms of Legendre scaling function is as follows

$$A(t) = \sum_{i=0}^{2^{K-1}} \sum_{j=0}^{M} A_{ij} \Phi_{ij}$$

Equation (5.8) can be written in vector form as follows.

$$A(t) = \left[A_{10}\ \ A_{11} \dots A_{1M}\ A_{2^{K-1}0} \dots A_{2^{K-1}M}\right]\Phi(t) \tag{5.8}$$

where A_{ij} is $s \times s$ constant matrix of the coefficients of Legendre scaling function $\Phi_{ij}(t)$. These matrices can be found by the following formula

$$A_{ij} = \int_{\frac{2i-2}{2^K}}^{\frac{2i}{2^K}} A(t)\Phi_{ij}(t)dt \tag{5.9}$$

In similar manner, $B(t)$ can be expand in terms of Legendre scaling function is as follows.

$$B(t) = \sum_{i=0}^{2^{K-1}} \sum_{j=0}^{M} B_{ij} \Phi_{ij}$$

also equation (5. 10) can be written in vector form as follows

$$B(t) = \left[B_{10}\ \ B_{11} \dots B_{1M}\ \ B_{2^{K-1}0} \dots B_{2^{K-1}M}\right]\Phi(t) \tag{5.10}$$

where B_{ij} is s x r constant matrix of the coefficients of Legendre scaling function $\Phi_{ij}(t)$. These matrices can be found by the following formula

$$B_{ij} = \int_{\frac{2i-2}{2^K}}^{\frac{2i}{2^K}} B(t)\Phi_{ij}(t)dt \tag{5.11}$$

5.3.2. Parameterization $A(t)$ and $B(t)$ of Chebyshev Scaling Function

The parameterization of time varying matrices $\mathbf{A(t)}$and $\mathbf{B(t)}$ from the problem of equation (1.4) can be expressed in terms of Chebyshev scaling functions. The approximation of can be given by

$$A(t) = \sum_{i=0}^{2^{K-1}} \sum_{j=0}^{M} A_{ij}\Psi_{ij}$$

where A_{ij} is $s \times s$ constant matrix of the coefficients of Legendre scaling function.

$\Psi_{ij}(t)$. These matrices can be found by the following formula

$$A(t) = \left[A_{10}\ \ A_{11} \ldots A_{1M}\ \ A_{2^{K-1}0} \ldots A_{2^{K-1}M}\right]\Psi(t) \tag{5.14}$$

$$A_{ij} = \int_{\frac{i-1}{2^{K-1}}}^{\frac{i}{2^{K-1}}} A(t)\Psi_{ij}(t)dt \tag{5.15}$$

In similar manner, $B(t)$ can be expand in terms of Chebyshev scaling functions as follows

$$B(t) = \sum_{i=0}^{2^{K-1}} \sum_{j=0}^{M} B_{ij}\Psi_{ij}$$

also equation (5.16) can be written in vector form as follows

$$B(t) = \left[B_{10}\ \ B_{11} \ldots B_{1M}\ \ B_{2^{K-1}0} \ldots B_{2^{K-1}M}\right]\Psi(t) \tag{5.16}$$

where B_{ij} is s x r constant matrix of the coefficients of Chebyshev scaling functions

$\Psi_{ij}(t)$. These matrices can be found by the following formula

$$B_{ij} = \int_{\frac{i-1}{2^{K-1}}}^{\frac{i}{2^{K-1}}} B(t)\Psi_{ij}(t)dt \tag{5.17}$$

We can construct the quadratic form of linear time varying optimal problems from the integration of state equation (1.4), equation (1.3) can be rewritten using Legendre and

Chebyshev scaling function with operational matrix of integration, and unknown coefficients to be easily solved using MATLAB in the following form.

Now to express the state equations in terms of the unknown parameters of the state variables and the control variables, equation (1.4) can be integrated as

$$x(t) - x(0) = \int_0^t A(t)x(\tau)d\tau + \int_0^t B(t)u(\tau)d\tau \qquad (5.18)$$

By substituting equation (5.2), (5.8), (5.10) and initial vector into (5.18)

$$\left(I_s \otimes \Phi^T(t)\right) a - \left(I_s \otimes \Phi^T(t)\right) \eth a^0$$
$$= \int_0^t [A_{10}\ \ A_{11} \ldots A_{1M}\ \ A_{2^{K-1}0} \ldots A_{2^{K-1}M}] \Phi(t) \left(I_s \otimes \Phi^T(t)\right) a dt$$
$$+ \int_0^t [B_{10}\ \ B_{11} \ldots B_{1M}\ \ B_{2^{K-1}0} \ldots B_{2^{K-1}M}] \Phi(t) \left(I_r \otimes \Phi^T(t)\right) b dt \qquad (5.19)$$

But

$$\left.\begin{array}{l} [A_{10}\ \ A_{11} \ldots A_{1M}\ \ A_{2^{K-1}0} \ldots A_{2^{K-1}M}] \Phi(t)\Phi^T(t) = \Phi^T(t)\widetilde{A} \\ [B_{10}\ \ B_{11} \ldots B_{1M}\ \ B_{2^{K-1}0} \ldots B_{2^{K-1}M}] \Phi(t)\Phi^T(t) = \Phi^T(t)\widetilde{B} \end{array}\right\} \qquad (5.20)$$

where $\widetilde{A}$ and $\widetilde{B}$are $sN \times sN$ and $sN \times rN$ constant matrices respectively. substituting equation (5.20) into equation (5.19)

$$\left(I_s \otimes \Phi^T(t)\right) a - \left(I_s \otimes \Phi^T(t)\right) \eth a^0$$
$$= \int_0^t \Phi^T(t)\widetilde{A} a dt + \int_0^t \Phi^T(t)\widetilde{B} b dt \qquad (5.21)$$

Now using the integration operational matrix P of Legendre and Chebyshev scaling function. We get

$$\left(I_s \otimes \Phi^T(t)\right) a - \left(I_s \otimes \Phi^T(t)\right) \eth a^0 = \Phi^T(t)(P^T \otimes I_s)\widetilde{A} a + \Phi^T(t)(P^T \otimes I_s)\widetilde{B} b$$

Using the Kronecker product properties and equating the coefficient of $I_s \otimes \Phi^T(t)$

$$I_{Ns} a - \eth a^0 = (P^T \otimes I_s)\widetilde{A} a + (P^T \otimes I_s)\widetilde{B} b$$

$$[(P^T \otimes I_s)\widetilde{A} - I_{Ns} \quad (P^T \otimes I_s)\widetilde{B}] \begin{bmatrix} a \\ b \end{bmatrix} = -\eth a^0 \qquad (5.22)$$

Combining the equality constraints (5.22) with (4.6) and (4.15) we get

$$\begin{bmatrix} (P^T \otimes I_s)\widetilde{A} - I_{Ns} & (P^T \otimes I_s)\widetilde{B} \\ (\Psi' \otimes I_s) & o_{(2^{K-1}-1)s \times Nr} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -\eth a^0 \qquad (5.23)$$

Then compact quadratic form using approximation of performance index and (5.23) as follows:

$$\left.\begin{array}{l} \min_z \quad z^T H z \\ \text{Subject to} \quad Fz = h \end{array}\right\} \tag{5.24}$$

$$\text{where, } z = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$H = \begin{bmatrix} RR \otimes Q & O_{Ns\times Nr} \\ O_{Nr\times Ns} & RR \otimes R \end{bmatrix}$$

$$F = \begin{bmatrix} (P^T \otimes I_s)\widetilde{A} - I_{Ns} & (P^T \otimes I_s)\widetilde{B} \\ (\Phi' \otimes I_s) & o_{(2^{K-1}-1)_{s\times Nr}} \end{bmatrix}$$

$$h = \begin{bmatrix} -g_0 \delta \\ o_{(2^{K-1}-1)_{s\times 1}} \end{bmatrix}$$

where $\widetilde{A}$ and $\widetilde{B}$ are the approximation of $A(t)$ and $B(t)$ from equation (5.14) and (5.16) respectively.

The general form of the linear time varying optimal control problems are equation (1.3) with quadratic performance indices and linear constraints. Then the general steps to approximate state variables, control variables and performance indices using Legendre and Chebyshev scaling function with unknown coefficients.

Step (1): Define the unknown coefficients of state and control variables according to K and M. The unknown coefficients will be as follows

$$a = [a_{10}\ a_{11} \ldots a_{1M}\ a_{20}\ a_{21} \ldots a_{2M} \ldots a_{2^{K-1}0} \ldots a_{2^{K-1}M}\ ]^T$$

$$b = [b_{10}\ \ b_{11} \ldots b_{1M}\ \ b_{20}\ b_{21} \ldots\ b_{2M}\ \ldots\ b_{2^{K-1}0} \ldots b_{2^{K-1}M}]^T$$

Step (2): Generate the scaling function depending on K and M.

$$\Phi(t) = [\Phi_{10}(t)\ \ \Phi_{11}(t) \ldots \Phi_{1M}(t)\ \Phi_{20}(t)\ \Phi_{21}(t) \ldots \Phi_{2M}(t) \ldots\ \Phi_{2^{K-1}0}(t) \ldots\ \ \Phi_{2^{K-1}M}(t)]^T$$

Step (3): Approximate the state and control variables.

$$x(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} a_{nm} \Phi_{nm}$$

$$u(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M} b_{nm} \Phi_{nm}$$

Step (4): Find the vector of initial condition

$$\delta = \frac{\sqrt{2}}{2^{K/2}}, \qquad \xi_0 = [x_i(0)\ 0\ 0 \ldots 0\ |x_i(0)\ 0\ 0 \ldots 0]^T$$

Step (5): Find the points of ensured continuity and the continuity matrix.

$$\Phi' = \left[\Phi_{10}(t_i) \ldots\ \Phi_{1M}(t_i)\ \Phi_{20}(t_i)\ \ldots\ \Phi_{2M}(t_i) \ldots\ \Phi_{2^{K-1}0}(t_i) \ldots\ \ \Phi_{2^{K-1}M}(t_i)\right]$$

Step (6): Find the approximation of $A(t)$ using equation (5.9) and (5.15) and $B(t)$ using (5.11) and (5.17). We get the equivalent quadratic programming problem.

Step (7): Solve for the quadratic programming problem using MATLAB.

Step (8): Substitute the unknown coefficient value from step (7) into step (3) we get the control variable and state variable.

Step (9): Substitute the unknown coefficient value from step (7) into the objective function, and we get the approximate value.

Example 5.3 (Jaddu, 2006**)**

Find the optimal control $u^*(t)$ and state variable $x^*(t)$ which minimize the performance index.

$$\min \qquad J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt$$

$$\text{Subject to } x'(t) = tx(t) + u(t) \quad , \quad x(0) = 1$$

Solution:

This optimal control problem transform into quadratic programming problem is similar steps to the above example 5.1 with the same scaling parameter (K=2).

Using Legendre scaling function for M=1

First, we approximate matrix A and matrix B using equation (5.9) and (5.11) respectively.

Then the equivalent quadratic programming problem is

$$\text{Min } \tfrac{1}{2}(a_{10}^2 + a_{11}^2 + a_{20}^2 + a_{21}^2 + b_{10}^2 + b_{11}^2 + b_{20}^2 + b_{21}^2)$$

$$\text{Subject to} \qquad \tfrac{-23}{24}a_{10} + \tfrac{1}{4}b_{10} - \tfrac{1}{4\sqrt{3}}b_{11} = \tfrac{-1}{\sqrt{2}}$$

$$\frac{\sqrt{3}}{48}a_{10} - \frac{47}{48}a_{11} + \frac{1}{4\sqrt{3}}b_{10} = 0$$

$$\frac{1}{8}a_{10} + \frac{\sqrt{3}}{24}a_{11} - \frac{5}{6}a_{20} - \frac{\sqrt{3}}{24}a_{21} + \frac{1}{2}b_{10} + \frac{1}{4}b_{20} - \frac{1}{4\sqrt{3}}b_{21} = \frac{-1}{\sqrt{2}}$$

$$\frac{\sqrt{3}}{16}a_{20}-\frac{47}{48}a_{21}+\frac{1}{4\sqrt{3}}b_{20}=0$$

$$1.4142a_{10}+2.4494a_{11}-1.4142a_{20}+2.4494a_{21}=0$$

For M=2 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2}(a_{10}^2+a_{11}^2+a_{12}^2+a_{20}^2+a_{21}^2+a_{22}^2+b_{10}^2+b_{11}^2+b_{12}^2+b_{20}^2+b_{21}^2+b_{22}^2)$$

Subject to $\frac{1}{24}a_{10}-\frac{1}{12\sqrt{3}}a_{11}+\frac{1}{8\sqrt{20}}a_{11}+\frac{1}{4}b_{10}-\frac{1}{4\sqrt{3}}b_{11}=\frac{-1}{\sqrt{2}}$

$$\frac{1}{24\sqrt{3}}a_{10}-a_{11}+\frac{1}{4\sqrt{15}}a_{12}+\frac{1}{4\sqrt{3}}b_{10}+\frac{-1}{4\sqrt{15}}b_{12}=0$$

$$\frac{1}{48\sqrt{5}}a_{11}+\frac{1}{4\sqrt{15}}b_{11}=0$$

$$\frac{1}{2}a_{10}+\frac{\sqrt{3}}{6}a_{11}+\frac{\sqrt{5}}{6}a_{12}+\frac{1}{6}a_{20}-\frac{1}{16\sqrt{3}}a_{21}-\frac{1}{12\sqrt{20}}a_{22}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}+\frac{-1}{4\sqrt{3}}b_{21}=\frac{-1}{\sqrt{2}}$$

$$\frac{3}{16\sqrt{3}}a_{20}+\frac{3}{16\sqrt{15}}a_{21}-\frac{3}{16\sqrt{15}}a_{22}+\frac{1}{4\sqrt{3}}b_{20}+\frac{-1}{4\sqrt{15}}b_{22}=0$$

$$\frac{1}{48\sqrt{5}}a_{20}+\frac{3}{16\sqrt{15}}a_{21}+\frac{1}{120}a_{22}+\frac{1}{4\sqrt{15}}b_{21}=0$$

$$1.4142a_{10}+2.4494a_{11}+3.1622a_{12}-1.4142a_{20}+2.4494a_{21}-3.1622a_{22}=0$$

For M=3 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2}(a_{10}^2+a_{11}^2+a_{12}^2+a_{13}^2+a_{20}^2+a_{21}^2+a_{22}^2+a_{23}^2+b_{10}^2+b_{11}^2+b_{12}^2+b_{13}^2+b_{20}^2+b_{21}^2+b_{22}^2+b_{23}^2)$$

Subject to $\frac{1}{24}a_{10}-\frac{1}{12\sqrt{3}}a_{12}-\frac{3\sqrt{5}}{70}a_{13}+\frac{1}{4}b_{10}-\frac{1}{4\sqrt{3}}b_{11}=\frac{-1}{\sqrt{2}}$

$$\frac{-1}{16\sqrt{3}}a_{10}+\frac{1}{12\sqrt{5}}a_{11}+\frac{1}{8\sqrt{3}}a_{12}+\frac{1}{4\sqrt{3}}b_{10}-\frac{1}{4\sqrt{15}}b_{12}=0$$

$$\frac{1}{48\sqrt{5}}a_{10}+\frac{1}{16\sqrt{15}}a_{12}-\frac{27}{14\sqrt{35}}a_{12}-\frac{6}{280\sqrt{3}}a_{13}+\frac{1}{4\sqrt{15}}b_{11}-\frac{1}{4\sqrt{35}}b_{13}=0$$

$$\frac{1}{8\sqrt{7}}a_{10}+\frac{\sqrt{3}}{12\sqrt{35}}a_{12}+\frac{27}{112\sqrt{35}}a_{12}+\frac{3}{56\sqrt{35}}a_{13}+\frac{1}{4\sqrt{35}}b_{12}=0$$

$$\frac{1}{2}a_{10}+\frac{\sqrt{3}}{6}a_{11}+\sqrt{3}a_{12}+\frac{1}{24}a_{20}+\frac{\sqrt{3}}{48}a_{21}-\frac{1}{6\sqrt{20}}a_{22}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}-\frac{1}{4\sqrt{3}}b_{21}=\frac{-1}{\sqrt{2}}$$

$$\frac{3}{12\sqrt{3}}a_{20}+\frac{1}{4\sqrt{8}}a_{21}-\frac{3}{16\sqrt{15}}a_{22}-\frac{3}{140\sqrt{15}}a_{22}+\frac{1}{4\sqrt{3}}b_{20}-\frac{1}{4\sqrt{15}}b_{22}=0$$

$$\frac{1}{48\sqrt{5}}a_{20}+\frac{3}{6\sqrt{15}}a_{21}+\frac{1}{120}a_{22}+\frac{1}{4\sqrt{15}}b_{21}-\frac{1}{4\sqrt{35}}b_{23}=0$$

$$\frac{\sqrt{3}}{210\sqrt{7}}a_{21}+\frac{3}{16\sqrt{35}}a_{22}+\frac{3}{140\sqrt{35}}a_{23}+\frac{1}{4\sqrt{35}}b_{22}=0$$

$$1.4142a_{10}+2.4494a_{11}+3.1622a_{12}+3.7416\,a_{13}-1.4142a_{20}+2.4494a_{21}-3.1622a_{22}+3.7416a_{23}=0$$

Using Chebyshev scaling function For M=1

First, we approximate matrix A and matrix B using equation (5.15) and (5.17) respectively

Then the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2\pi}(2a_{10}^2+\frac{4}{3}a_{11}^2+2a_{20}^2+\frac{4}{3}a_{21}^2+2b_{10}^2+\frac{4}{3}b_{11}^2+2b_{20}^2+\frac{4}{3}b_{21}^2)$$

Subject to
$$\frac{5}{48\pi\sqrt{2}}a_{10}+\frac{1}{8\sqrt{2}}a_{11}+\frac{1}{4}b_{10}-\frac{1}{8\sqrt{2}}b_{11}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{16\pi}a_{10}+\frac{1}{24\pi\sqrt{2}}a_{11}+\frac{1}{4\sqrt{2}}b_{10}=0$$

$$\frac{\sqrt{2}}{8\pi}a_{10}+\frac{1}{16\pi}a_{11}+\frac{11}{48\pi\sqrt{2}}a_{20}+\frac{1}{24\pi}a_{21}+\frac{1}{2}b_{10}+\frac{1}{4}b_{20}-\frac{1}{8\sqrt{2}}b_{21}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{8\pi}a_{20}+\frac{1}{24\pi\sqrt{2}}a_{21}+\frac{1}{4\sqrt{2}}b_{20}=0$$

$$1.1284a_{10}+1.5958a_{11}-1.1284a_{20}+1.5958a_{21}=0$$

For M=2 the equivalent quadratic programming problem is

$$\text{Min}\ \frac{1}{2\pi}(2a_{10}^2-\frac{4\sqrt{2}}{3}a_{10}a_{12}+\frac{4}{3}a_{11}^2+0.594\pi a_{12}^2+2a_{20}^2-\frac{4\sqrt{2}}{3}a_{20}a_{22}+\frac{4}{3}a_{21}^2+0.594\pi a_{22}^2$$
$$+2b_{10}^2-\frac{4\sqrt{2}}{3}b_{10}b_{12}+\frac{4}{3}b_{11}^2+0.594\pi b_{12}^2+2b_{20}^2-\frac{4\sqrt{2}}{3}b_{20}b_{22}+\frac{4}{3}b_{21}^2+0.594\pi b_{22}^2)$$

Subject to
$$\frac{5}{48\pi\sqrt{2}}a_{10}-\frac{1}{18\pi\sqrt{2}}a_{11}-\frac{1}{12\pi\sqrt{10}}a_{12}-\frac{9}{8\pi}b_{10}+\frac{11}{6\pi\sqrt{2}}b_{11}+\frac{23}{8\pi\sqrt{2}}b_{12}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{12\pi}a_{10}+\frac{\sqrt{5}}{24\pi\sqrt{2}}a_{11}-\frac{\sqrt{2}}{24\pi}a_{12}-\frac{11}{8\pi\sqrt{2}}b_{10}+\frac{1}{4\pi\sqrt{5}}b_{11}+\frac{23}{16\pi}b_{12}=0$$

$$\frac{7}{96\pi}a_{10}+\frac{1}{24\pi}a_{11}+\frac{1}{3\sqrt{20}}a_{11}+\frac{11\sqrt{2}}{8\pi}b_{10}+\frac{1}{16\pi\sqrt{5}}b_{11}+\frac{3}{2\pi\sqrt{5}}b_{12}=0$$

$$\frac{11\sqrt{2}}{36\pi}a_{10}+\frac{\sqrt{2}}{6}a_{12}-\frac{5}{4}a_{20}++\frac{1}{6\sqrt{2}}a_{22}-\frac{7}{4\pi}b_{10}+\frac{1}{6\pi\sqrt{5}}b_{11}+\frac{17\sqrt{2}}{12\pi}b_{12}-\frac{1}{6\sqrt{2}}b_{21}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{11}{16\pi}a_{20}+\frac{1}{24\pi\sqrt{20}}a_{21}-\frac{5}{4\pi\sqrt{2}}a_{22}+\frac{11}{24\pi\sqrt{2}}b_{20}+\frac{1}{2\pi\sqrt{5}}b_{21}+\frac{5}{24\pi}b_{21}=0$$

$$\frac{1}{12\pi}a_{20}+\frac{1}{2\pi\sqrt{2}}a_{21}+\frac{1}{3\pi\sqrt{10}}a_{22}+\frac{\sqrt{2}}{32\pi}b_{20}+\frac{3\sqrt{2}}{4\pi\sqrt{5}}b_{21}+\frac{\sqrt{2}}{\pi\sqrt{5}}b_{22}=0$$

$$1.1284a_{10}+1.5958a_{11}+1.5958a_{12}-1.1284a_{20}+1.5958a_{21}-1.5958a_{22}=0$$

For M=3 the equivalent quadratic programming problem is

Min $\frac{1}{2\pi}(2a_{10}^2-\frac{4\sqrt{2}}{3}a_{10}a_{12}+\frac{4}{3}a_{11}^2+\frac{28}{15}a_{12}^2+\frac{8}{5}a_{11}a_{13}+\frac{68}{35}a_{13}^2+2a_{20}^2-\frac{4\sqrt{2}}{3}a_{20}a_{22}+\frac{4}{3}a_{21}^2+\frac{28}{15}a_{22}^2+\frac{8}{5}a_{21}a_{23}+\frac{68}{35}a_{23}^2+2b_{10}^2-\frac{4\sqrt{2}}{3}b_{10}b_{12}+\frac{4}{3}b_{11}^2+\frac{28}{15}b_{12}^2+\frac{8}{5}b_{11}b_{13}+\frac{68}{35}b_{13}^2+2b_{20}^2-\frac{4\sqrt{2}}{3}b_{20}b_{22}+\frac{4}{3}b_{21}^2+\frac{28}{15}b_{22}^2+\frac{8}{5}b_{21}b_{23}+\frac{68}{35}b_{23}^2$

Subject to $\frac{1}{8\sqrt{2}}a_{10}-\frac{1}{144\pi}a_{11}-\frac{7}{72\pi}a_{12}-\frac{1}{40\pi}a_{13}+\frac{5}{9\pi\sqrt{2}}b_{10}-\frac{1}{9\pi}b_{11}-\frac{1}{8\pi\sqrt{2}}b_{12}+$

$$\frac{1}{12\pi}b_{13}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{1}{12\pi}a_{10}-\frac{1}{48\pi\sqrt{2}}a_{11}-\frac{1}{24\pi\sqrt{2}}a_{12}+\frac{1}{3\pi}b_{10}-\frac{5}{12\pi\sqrt{2}}b_{12}=0$$

$$-\frac{1}{96\pi}a_{10}-\frac{1}{96\pi\sqrt{2}}a_{11}+\frac{\sqrt{2}}{64\pi}a_{12}-\frac{1}{96\pi}a_{13}-\frac{\sqrt{2}}{16\pi}b_{11}+\frac{1}{16\pi}b_{12}-\frac{\sqrt{2}}{48\pi}b_{13}=0$$

$$\frac{-1}{144\pi}a_{10}+\frac{1}{144\pi\sqrt{2}}a_{11}+\frac{1}{48\pi\sqrt{2}}a_{12}+\frac{1}{144\pi\sqrt{2}}a_{13}-\frac{1}{36\pi}b_{10}+\frac{\sqrt{2}}{24\pi}b_{12}=0$$

$$\frac{11\sqrt{2}}{72\pi}a_{10}+\frac{1}{18\pi}a_{11}-\frac{1}{6\pi}a_{12}-\frac{27}{36\pi}a_{13}+\frac{7}{96\pi}a_{21}-\frac{9}{8\pi}a_{22}-\frac{1}{70\pi}a_{23}+\frac{7\sqrt{2}}{18\pi}b_{10}-\frac{2}{3\pi}b_{12}+$$

$$\frac{5}{9\pi\sqrt{2}}b_{20}-\frac{1}{8\pi}b_{21}+\frac{1}{8\pi\sqrt{2}}b_{22}-\frac{1}{8\pi}b_{23}=\frac{-\sqrt{\pi}}{2}$$

$$\frac{5}{4\pi}a_{20}+\frac{1}{48\pi\sqrt{2}}a_{21}-\frac{5}{4\pi\sqrt{2}}a_{22}-\frac{1}{40\pi\sqrt{2}}a_{23}+\frac{1}{3\pi}b_{20}-\frac{5}{12\pi\sqrt{2}}b_{22}=0$$

$$\frac{1}{96\pi}a_{20}+\frac{\sqrt{2}}{32\pi}a_{21}+\frac{\sqrt{2}}{192\pi}a_{22}-\frac{9\sqrt{2}}{64\pi}a_{23}+\frac{\sqrt{2}}{16\pi}b_{20}+\frac{1}{16\pi}b_{22}-\frac{\sqrt{2}}{48\pi}b_{23}=0$$

$$\frac{-3}{8\pi}a_{20}+\frac{1}{144\pi\sqrt{2}}a_{21}+\frac{\sqrt{2}}{48\pi}a_{22}+\frac{\sqrt{2}}{288\pi}a_{23}-\frac{1}{36\pi}b_{20}+\frac{\sqrt{2}}{24\pi}b_{22}=0$$

$$1.1284a_{10}+1.5958a_{11}+1.5958a_{12}+1.5958a_{13}-1.1284a_{20}+1.5958a_{21}-1.5958a_{22}-1.5958a_{23}=0$$

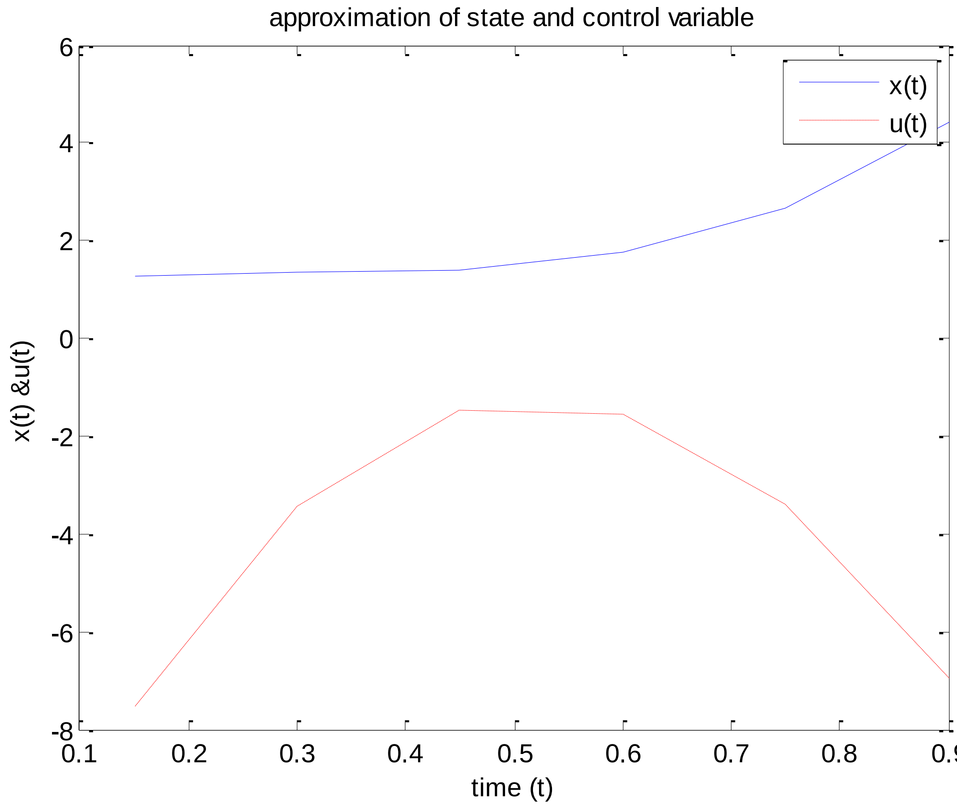

Figure 8. The approximation of the state and control variable using Legendre scaling function

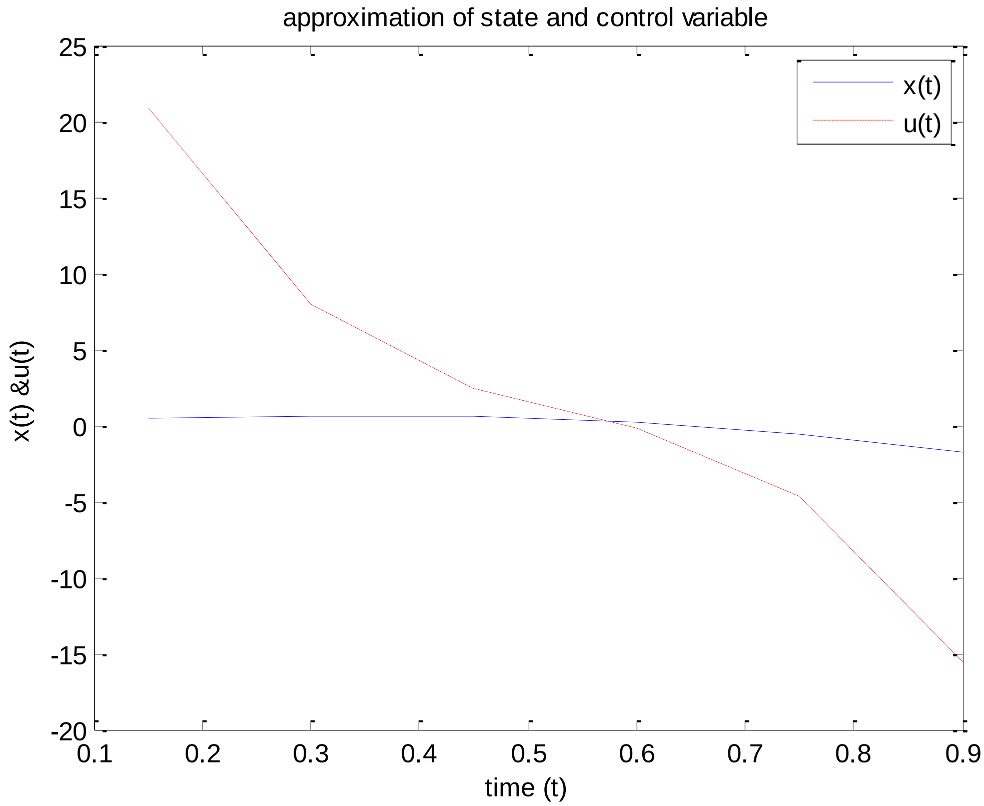

Figure 9. The approximation of state and control variable using Chebyshev scaling function

These Figure 8 and 9 are shows the approximation value of the state variable x(t) and the control variable u(t) at a time t, t∈ (0,1). The results were obtained increase the degree of Legendre and Chebyshev scaling function it is noted that the control variable and state variable is not monotonic (i.e. neither increases nor decreases) when we increase time interval. Such behavior indicates that an increases time does not necessarily improve the estimate accuracy value of state variable and control variable.

Table 3. A comparison between Legendre and Chebyshev scaling function with the approximation of performance index of example 5.3.

	optimal value (Elnagar, 1997)		0.484290813333	
Order of poly	Performance index (J)			
	Legendre	Error	Chebyshev	Error
M=1	0.473005235031115	1.13×10^{-2}	0.421225626818946	6.30×10^{-2}
M=2	0.485846388354424	1.55×10^{-3}	0.474117777216600	2.01×10^{-3}
M=3	0.484919052338766	6.28×10^{-4}	0.484944922800065	6.54×10^{-4}

In Table (3), shows that the approximation value of performance index for the linear time varying quadratic OCP using Legendre scaling function and Chebyshev scaling function; when we applied theses methods at example (5.3) we see that the effectiveness of the Legendre scaling function method from Chebyshev scaling function. This shows that increasing the degree of polynomial (M), the performance index (J) moves closer to the optimal value. The programming was performed using MATLAB 8.1 to execute the program.

5.4. Advantages and Disadvantages of Legendre and Chebyshev scaling Function Methods

Legendre and Chebyshev scaling function Methods have several advantages such as:

1. Easy approximation of state variable and control variable,
2. No integration of the state equations,
3. No need to use co-state variables (adjoint function),
4. Explicit formula was derive to approximate the quadratic performance index and
5. Small quadratic programming problems were to be solved.
6. To their disadvantage, Legendre and Chebyshev scaling function methods defined only the time interval between zero and one ($\text{i. e } t \in [0,1]$).

6. SUMMARY, CONCLUSION AND RECOMMENDATION

6.1. Summary

Optimal control problem is the process of determining control variables and state variables for a dynamic system as a function of time to minimize (maximize) the performance indices. Linear quadratic optimal control problem is a special type of optimal control problem that deals with linear system and minimization of cost or objective functions that are quadratic in state and in control (subject to some constraints) with performance index determination. This project works depends on state-control parameterization technique via Legendre scaling function and Chebyshev scaling function in which the linear state equations are replaced by a sequence of both LTI and LTV algebraic equations. The aims of the proposed methods are to approximate the control variables, state variables and performance index. In order to solve linear time quadratic optimal control systems by using Legendre scaling function and Chebyshev scaling function methods are firstly changed the linear time quadratic OCPs into QP problem, then the QP problems was solved using MATLAB code.

In addition, we choose Legendre scaling function and Chebyshev scaling function, which have many advantages over the other orthogonal polynomials, and functions. These two methods have several advantages such as easy approximation, no integration of the state equations, no need to use co-state variables (Lagrange multiplier), explicit formula was derive to approximate the quadratic performance index, and small quadratic programming problems were to be solved. These scaling functions approximate the control and/or state by finite terms and employing an OMI to eliminate the integral operations. The proposed methods are suitable for solving both LTI and LTV quadratic optimal control systems.

6.2. Conclusion

In this project, we proposed numerical methods to solve different types of optimal control problems such as both LTI and LTV quadratic optimal control systems by using Legendre scaling function and Chebyshev scaling function methods. These two methods were based on parameterizing the system of state variables and control variables using a finite length of scaling functions. Applying these methods, the linear time quadratic OCPs was converted into QP problems with in unknown coefficients and known scaling function and it has solved the

QP problems using MATLAB. Furthermore, the proposed methods could be easily implemented in a MATLAB. The results obtained for the linear time optimal control systems with quadratic performance index using Legendre and Chebyshev scaling functions works for finding the approximation values of the state variable x(t), the control variable u(t) and performance indices. It was also deduced that the polynomial of scaling function with degree three and the time interval $t \in [0,1]$, the absolute error is less (almost zero) with the Legendre scaling functions than with the Chebyshev scaling functions method. Hence, the Legendre scaling function method is more suitable for solving the linear time quadratic optimal control systems.

6.3. Recommendation

Based on linear time quadratic optimal control systems the following basic recommendations are suggested:

- In control-state parameterization with large number of unknown coefficients, which have to be determined a_i and b_i; the system state equations have to be replace by a large number of equality constraints; so control parameterization can be use instead of control-state parameterization.
- Solve nonlinear time quadratic optimal control problems with inequality constraints using these two methods.
- Use operational matrix of differentiation instead of operational matrix of integration and solve the problem using Legendre scaling function and Chebyshev scaling function.
- Solve linear time quadratic optimal control problems using these two methods with the time interval t out of this range i.e. $t \notin [0,1]$ and the degree of polynomial is greater than three.

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8. APPENDIX

%%%%%%%MATLAB code to compute the quadratic programming problems using Legendre scaling function and Chebyshev scaling function.

%%%%%% For example 5.1

%% write the Legendre scaling functions in function. m

```
function [L10 ] = L10( t )
t=0.15:0.15:1;
L10=sqrt(2);
end
function [ L11 ] = L11( t )
t=[0.15:0.15:1];
L11=sqrt(6)*(4*t-1);
end
function [ L12 ] = L12( t )
t=[0.15:0.15:1];
L12=sqrt(5/2)*(3*(4*t-1).^2-1);
 end
function [ L13] = L13( t )
t=[0.15:0.15:1];
L13=sqrt(7/2)*(5*(4*t-1).^3-3*(4*t-1));
end
function [ L21 ] = L21( t )
t=[0.15:0.15:1];
L21=sqrt(6)*(4*t-3);
end
 function [ L22 ] = L22( t )
t=[0.15:0.15:1];
L22=sqrt(5/2)*(3*(4*t-3).^2-1);
end
function [ L23] = L23( t )
t=[0.15:0.15:1];
```

```
L23=sqrt(7/2)*(5*(4*t-3).^3-3*(4*t-3));
end
%%%%Legendre scaling functions
%%% END
%% write the coefficient matrices of quadratic programming problem in m. file
clear all % clears previously declared symbols.
clc % Clear command window
%% Now calling Legendre scaling functions
format long
z=[16,1];
U=zeros(4,4);
U(1,1)=2;
diag= zeros(4,4);
diag(1,2)=1/sqrt(3);
diag(2,1)=(-1)/sqrt(3);
diag(2,3)=1/sqrt(15);
diag(3,2)=(-1)/sqrt(15);
diag(3,4)=1/sqrt(35);
diag(4,3)=(-1)/sqrt(35);
diag(1,1)=1;
H=eye(16); % 16 by 16 identity matrix (hessian matrix)
O=zeros(4,4); % 4 by 4 zero matrix
I=eye(8,8); % 8 by 8 identity matrix
P=(1/4)*[diag U; O diag]; % the operational matrix of integration
tranP=transpose(P); % the transpose of  operational matrix of integration
d=(-1)*tranP;
o=zeros(1,8); % 1 by 8 matrix
f=d-I;
c=[1.4142  2.4494 3.1622 3.7417 -1.4142 2.4494  -3.1622 3.7417];% continuity matrix
F=[f  tranP; c o]; % Matrix in linear equality constraints F*z = h
h=[(-1)*sqrt(2)/2 0 0 0 (-1)*sqrt(2)/2 0 0 0 0]'; %vector for linear equality constraints
a = zeros(1,16);    % Vectors in linear term a*x
```

```
Fineq = [ ];     % Matrix in linear inequality constraints F*z = h
hineq= [ ];     % Vector in linear inequality constraints F*z = h
lb = [ ];   % Vector of lower bounds
ub = [ ];   % Vector of lower bounds
x0 = 1;   % Initial point of x
opts=optimoptions('quadprog','Algorithm','interior-point-convex','Display','off'); % minimizes
with the default optimization parameters replaced by values in options, an argument created
with the optimoptions function.
[z ,J]=quadprog (H, a, F, h, Fineq, hineq, lb, ub, x0, opts);
display(z)
display(J)
a10=z(1);a11=z(2);a12=z(3);a13=z(4);a20=z(5);a21=z(6);a22=z(7);a23=z(8);
b10=z(9);b11=z(10);b12=z(11);b13=z(12);b20=z(13);b21=z(14);b22=z(15);b23=z(16);
X=a10*L10+a11*L11+a12*L12+a13*L13+a20*L10+a21*L21+a22*L22+a23*L23
U=b10*L10+b11*L11+b12*L12+b13*L13+b20*L10+b21*L21+b22*L22+b23*L23
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%
%%%%MATLAB code using Chebyshev scaling function
%%%write the Chebyshev scaling function in function.m
function [ C10 ] = C10( t )
t=[0.15:0.15:1];
C10=2/sqrt(pi);
end
function [C11 ] = C11( t )
t=0.15:0.15:1;
C11=(2*sqrt(2)/sqrt(pi))*(4*t-1);
end
function [ C12] = C12(t )
t=0.15:0.15:1;
C12=(2*sqrt(2)/sqrt(pi))*(2*(4*t-1).^2-1);
end
```

```
function [ C13 ] = C13( t )
t=0.15:0.15:1;
C13=(2*sqrt(2)/sqrt(pi))*(4*(4*t-1).^3-3*(4*t-1));
end
function [ C21] = C21( t )
t=0.15:0.15:1;
C21=(2*sqrt(2)/sqrt(pi))*(4*t-3);
end
function [ C22 ] = C22( t )
t=0.15:0.15:1;
C22=(2*sqrt(2)/sqrt(pi))*(2*(4*t-3).^2-1);
end
function [ C23 ] = C23( t )
t=0.15:0.15:1;
C23=(2*sqrt(2)/sqrt(pi))*(4*(4*t-3).^3-3*(4*t-3));
end
%%%% Chebyshev scaling functions
%%END
%%  write the coefficient matrices of quadratic programming problem in m. file
clear all % clears previously declared symbols.
clc % Clear command window
%%%% Now calling the Chebyshev scaling functions
format long
z=[16,1];
diag= zeros(4,4);
diag(1,1)=1/4;
diag(1,2)=1/(4*sqrt(2));
diag(2,1)=(-1)/(8*sqrt(2));
diag(3,1)=(-1)/(6*sqrt(2));
diag(2,3)=1/24;
diag(3,2)=(-1)/8;
```

```
diag(3,4)=1/24;
diag(4,1)=(-1)/(32*sqrt(2));
C=diag;
S=zeros(4,4);
S(1,1)=1/2;
S(3,1)=(-1)*sqrt(2)/6;
O=zeros(4,4);
P=[C S; O C];
tranP=transpose(P);
RR=[2/pi 0 (-2)*sqrt(2)/(3*pi) 0 0 0 0 0; 0 4/(3*pi) 0 -4/(5*pi)  0 0  0  0 ;
  (-2)*sqrt(2)/(3*pi)  0  28/(15*pi) 0 0 0 0 0; 0 -4/(5*pi) 0 68/(35*pi) 0 0 0 0;
 0 0 0 0 2/pi  0  (-2)*sqrt(2)/(3*pi) 0;0 0 0 0 0 4/(3*pi) 0 -4/(5*pi);
 0 0 0 0 (-2)*sqrt(2)/(3*pi)  0  28/(15*pi) 0;0 0 0 0 0 -4/(5*pi) 0 68/(35*pi)];
O1=zeros(8);
H=[RR O1;O1 RR];
I=eye(8,8);
d=(-1)*tranP;
o=zeros(1,8);
f=d-I;
c=[1.1284  1.5958  1.5958 1.5958   -1.1284 1.5958 -1.5958 -1.5958];% continuity matrix
F=[f tranP; c o]; % Matrix in linear equality constraints F*z = h
h=[(-1)*sqrt(pi)/2 0 0 0 (-1)*sqrt(pi)/2  0 0 0 0]'; %vector for linear equality constraints
a = zeros(1,16);    %Vector in linear term a'*x
Fineq = [ ];     % Matrix in linear inequality constraints F*z = h
hineq= [ ];     %Vector in linear inequality constraints F*z = h
lb = [ ];   %Vector of lower bounds
ub = [ ];   %Vector of lower bounds
x0 = 1;   %Initial point of  x
opts=optimoptions('quadprog','Algorithm','interior-point-convex','Display','off'); % minimizes
with the default optimization parameters replaced by values in options, an argument created
with the optimoptions function.
```

```
[z ,J]=quadprog(H, a, F, h, Fineq, hineq, lb, ub, x0, opts);
display(z)
display(J)
a10=z(1);a11=z(2);a12=z(3);a13=z(4);a20=z(5);a21=z(6);a22=z(7);a23=z(8);
b10=z(9);b11=z(10);b12=z(11);b13=z(12);b20=z(13);b21=z(14);b22=z(15);b23=z(16);
X= a10*C10+a11*C11+a12*C12+a13*C13+a20*C10+a21*C21+a22*C22+a23*C23
U= b10*C10+b11*C11+b12*C12+b13*C13+b20*C10+b21*C21+b22*C22+b23*C23
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%For example 5.2
%%%% MATLAB code using Legendre scaling function
%%  write the coefficient matrices of quadratic programming problem in m. file
clear all % clears previously declared symbols.
clc % Clear command window
%% %% Now calling Legendre scaling functions
format long
z=[16,1];
U=zeros(4,4);
U(1,1)=2;
diag= zeros(4,4);
diag(1,2)=1/sqrt(3);
diag(2,1)=(-1)/sqrt(3);
diag(2,3)=1/sqrt(15);
diag(3,2)=(-1)/sqrt(15);
diag(3,4)=1/sqrt(35);
diag(4,3)=(-1)/sqrt(35);
diag(1,1)=1;
Q=[2 0 0 0 0 0 0 0; 0 2 0 0 0 0 0 0; 0 0 2 0 0 0 0 0; 0 0 0 2 0 0 0 0;
   0 0 0 0 2 0 0 0; 0 0 0 0 0 2 0 0; 0 0 0 0 0 0 2 0; 0 0 0 0 0 0 0 2];
I1=eye(8); % 8 by 8 identity matrix
O1=zeros(8); % 8 by 8 zero matrix or all element are zeros
```

```
H=[Q O1;O1 I1]; % the Hessian matrix
O=zeros(4,4); % 4 by 4 zero matrix or all element are zeros
I=eye(8,8); % 4 by 4 identity matrix
P=(1/4)*[diag U; O diag]; % the operational matrix of integration
tranP=transpose(P); % the transpose of operational matrix of integration
d=(1/2)*tranP;
o=zeros(1,8); % 1 by 8 row matrix
f=d-I;
c=[1.4142 2.4494 3.1622 3.7417 -1.4142 2.4494  -3.1622 3.7417];% continuity matrix
F=[f  tranP; c  o]; % Matrix in linear equality constraints F*z = h
h=[(-1)*sqrt(2)/2 0 0 0 (-1)*sqrt(2)/2 0 0 0 0]'; % vector for linear equality constraints
a = zeros(1,16);    %Vector in linear term a'*x
Fineq = [ ];     % Matrix in linear inequality constraints F*z = h
hineq= [ ];     %Vector in linear inequality constraints F*z = h
lb = [ ];   %Vector of lower bounds
ub = [ ];   %Vector of lower bounds
x0 = 1;   %Initial point of  x
opts=optimoptions('quadprog','Algorithm','interior-point-convex','Display','off'); % minimizes
with the default optimization parameters replaced by values in options, an argument created
with the optimoptions function.
[z ,J]=quadprog(H, a, F, h, Fineq, hineq, lb, ub, x0, opts);
display(z)
display(J)
a10=z(1);a11=z(2);a12=z(3);a13=z(4);a20=z(5);a21=z(6);a22=z(7);a23=z(8);
b10=z(9);b11=z(10);b12=z(11);b13=z(12);b20=z(13);b21=z(14);b22=z(15);b23=z(16);
X=a10*L10+a11*L11+a12*L12+a13*L13+a20*L10+a21*L21+a22*L22+a23*L23
U=b10*L10+b11*L11+b12*L12+b13*L13+b20*L10+b21*L21+b22*L22+b23*L23
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% %%%MATLAB code using Chebyshev scaling function
%%  write the coefficient matrices of quadratic programming problem in m. file
clear all % clears previously declared symbols.
```

```
clc % Clear command window
%%%%% Now calling the Chebyshev scaling functions
format long
z=[16,1];
diag= zeros(4,4);
diag(1,1)=1/4;
diag(1,2)=1/(4*sqrt(2));
diag(2,1)=(-1)/(8*sqrt(2));
diag(3,1)=-1/(6*sqrt(2));
diag(2,3)=1/16;
diag(3,2)=-1/8;
diag(2,2)=0;
C=diag;
S=zeros(3,3);
S(3,1)=-sqrt(2)/6;
S(1,1)=1/2;
O=zeros(3,3);
P=[C S; O C];
tranP=transpose(P);
RR1=[4/pi  0  (-4)*sqrt(2)/(3*pi)  0 0  0 ;0 8/(3*pi)   0  0  0  0 ;
     (-4)*sqrt(2) /(3*pi)  0  1.188 0 0 0;0 0 0 4/pi  0  (-4)*sqrt(2)/(3*pi) ;
     0 0 0 0  8/(3*pi) 0; 0 0  0  (-4)*sqrt(2) /(3*pi)  0  1.188];
  RR2=[2/pi  0  (-2)*sqrt(2)/(3*pi)  0 0  0 ;0 4/(3*pi)   0  0  0  0 ;
     (-2)*sqrt(2) /(3*pi)  0  0.594 0 0 0;0 0 0 2/pi  0  (-2)*sqrt(2)/(3*pi) ;
     0 0 0 0  4/(3*pi) 0; 0 0  0  (-2)*sqrt(2) /(3*pi)  0  0.594];
  O1=zeros(6);
  H=[RR1 O1;O1 RR2];
I=eye(6,6);
d=1/2*tranP;
o=zeros(1,6);
f=d-I;
```

```
c=[1.1284  1.5958  1.5958   -1.1284 1.5958 -1.5958];% continuity matrix
F=[f  tranP; c o]; % Matrix in linear equality constraints F*z = h
h=[(-1)*sqrt(pi)/2 0 0 (-1)*sqrt(pi)/2  0 0 0]'; %vector for linear equality constraints
a = zeros(1,12);    %Vector in linear term a'*x
Fineq = [ ];     % Matrix in linear inequality constraints F*z = h
hineq= [ ];     %Vector in linear inequality constraints F*z = h
lb = [ ];   %Vector of lower bounds
ub = [ ];   %Vector of lower bounds
x0 = 1;   %Initial point of x
opts=optimoptions('quadprog','Algorithm','interior-point-convex','Display','off'); % minimizes
with the default optimization parameters replaced by values in options, an argument created
with the optimoptions function.
[z ,J]=quadprog(H, a, F, h, Fineq, hineq, lb, ub, x0, opts);
display(z)
display(J)
a10=z(1);a11=z(2);a12=z(3);a13=z(4);a20=z(5);a21=z(6);a22=z(7);a23=z(8);
b10=z(9);b11=z(10);b12=z(11);b13=z(12);b20=z(13);b21=z(14);b22=z(15);b23=z(16);
X= a10*C10+a11*C11+a12*C12+a13*C13+a20*C10+a21*C21+a22*C22+a23*C23
U= b10*C10+b11*C11+b12*C12+b13*C13+b20*C10+b21*C21+b22*C22+b23*C23
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%MATLAB code to compute the Legendre and Chebyshev
scaling function for the LTV quadratic optimal control problem.
%%%%%%%%%%%% For example 5.3
%%%% MATLAB code using Legendre scaling function
%%  write the coefficient matrices of quadratic programming problem in m. file
clear all % clears previously declared symbols.
clc % Clear command window
%% %% Now calling Legendre scaling functions
format long
z=[16,1];
```

```
U=zeros(4,4);
U(1,1)=2;
diag= zeros(4,4);
diag(1,2)=1/sqrt(3);
diag(2,1)=(-1)/sqrt(3);
diag(2,3)=1/sqrt(15);
diag(3,2)=(-1)/sqrt(15);
diag(3,4)=1/sqrt(35);
diag(4,3)=(-1)/sqrt(35);
diag(1,1)=1;
H=eye(16);
O=zeros(4,4);
I=eye(8,8);
P=(1/4)*[diag U; O diag];
tranP=transpose(P);
A1=[1/4 sqrt(3)/12 sqrt(5)/2 0 0 0 0 0;sqrt(3)/12  1/4  sqrt(3)/3*sqrt(20)  3*sqrt(15)/70 0 0 0 0;
   sqrt(5)/2 sqrt(3)/3*sqrt(20)  27/28  3/140  0 0 0 0;0 3*sqrt(15)/70 3/140  3*sqrt(5)/7 0 0 0 0;
   0 0 0 0  3/4   sqrt(3)/12 0 0; 0 0 0 0  sqrt(3)/12  (36+sqrt(60))/60  sqrt(3)/3*sqrt(20) 0;
   0 0 0 0 0  sqrt(3)/3*sqrt(20)  3/4  3/140;0 0 0 0 0 0 3/140 0];
%%%the coefficient matrix of  equality constraint
B1=eye(8);
d=tranP*A1;
d1=tranP*B1;
o=zeros(1,8);
f=d-I;
c=[1.4142  2.4494  3.1622  3.7417  -1.4142  2.4494  -3.1622  3.7417];% continuity matrix
F=[f  d1; c  o]; % Matrix in linear equality constraints F*z = h
h=[(-1)*sqrt(2)/2 0 0 0 (-1)*sqrt(2)/2 0 0 0 0]'; %vector for linear equality constraints
a = zeros(1,16);    %Vector in linear term a'*x
Fineq = [ ];     % Matrix in linear inequality constraints F*z = h
hineq= [ ];     %Vector in linear inequality constraints F*z = h
```

```
lb = [ ];   %Vector of lower bounds
ub = [ ];   %Vector of lower bounds
x0 = 1;   %Initial point of  x
opts=optimoptions('quadprog','Algorithm','interior-point-convex','Display','off'); % minimizes
with the default optimization parameters replaced by values in options, an argument created
with the optimoptions function.
[z ,J]=quadprog(H, a, F, h, Fineq, hineq, lb, ub, x0, opts);
display(z)
display(J)
a10=z(1);a11=z(2);a12=z(3);a13=z(4);a20=z(5);a21=z(6);a22=z(7);a23=z(8);
b10=z(9);b11=z(10);b12=z(11);b13=z(12);b20=z(13);b21=z(14);b22=z(15);b23=z(16);
X=a10*L10+a11*L11+a12*L12+a13*L13+a20*L10+a21*L21+a22*L22+a23*L23
U=b10*L10+b11*L11+b12*L12+b13*L13+b20*L10+b21*L21+b22*L22+b23*L23
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% %%%MATLAB code using Chebyshev scaling function
%%  write the coefficient matrices of quadratic programming problem in m. file
clear all % clears previously declared symbols.
clc % Clear command window
%%%%% Now calling the Chebyshev scaling functions
format long
z=[16,1];
diag= zeros(4,4);
diag(1,1)=1/4;
diag(1,2)=1/(4*sqrt(2));
diag(2,1)=(-1)/(8*sqrt(2));
diag(3,1)=-1/(6*sqrt(2));
diag(2,3)=1/16;
diag(3,2)=-1/8;
diag(3,4)=1/24;
diag(4,1)=(-1)/(24*sqrt(2));
C=diag;
```

```
S=zeros(4,4);
S(1,1)=1/2;
S(3,1)=(-1)*sqrt(2)/6;
O=zeros(4,4);
P=[C S; O C];
tranP=transpose(P);
RR1=[4/pi   0  (-4)*sqrt(2)/3*pi  0 0 0 0 0; 0  8/3*pi  0  -8/5*pi   0 0  0  0 ;
       (-4)*sqrt(2)/3*pi  0   1.188  0 0 0 0 0; 0  -8/5*pi  0  136/35*pi  0 0 0 0;
   0 0 0 0  4/pi   0   (-4)*sqrt(2)/3*pi 0;0 0 0 0 0  8/3*pi 0  -8/5*pi;
   0 0 0 0  (-4)*sqrt(2)/3*pi  0  1.188  0;0 0 0 0 0  -8/5*pi 0  136/35*pi];
 RR2=[2/pi  0   (-2)*sqrt(2)/(3*pi) 0 0 0 0 0; 0  4/(3*pi) 0  -4/(5*pi)  0 0  0  0 ;
       (-2)*sqrt(2)/(3*pi)  0   0.594  0 0 0 0 0; 0 -4/(5*pi)  0  68/(35*pi)  0 0 0 0;
   0 0 0 0  2/pi   0   (-2)*sqrt(2)/(3*pi)  0;0 0 0 0 0   4/(3*pi)  0  -4/(5*pi);
   0 0 0 0  (-2)*sqrt(2)/(3*pi)  0  0.594 0;0 0 0 0 0  -4/(5*pi)  0  68/(35*pi)];
   O1= zeros (8);
   H= [RR1 O1; O1 RR2];
A1= [sqrt(2)/(4*pi)  1/(6*pi)  -1/(6*pi)  -1/(10*pi)  0 0 0 0;1/(6*pi)  (sqrt(2)/(4*pi))-
sqrt(2)/(12*pi)   (sqrt(2)/(12*pi))-(sqrt(2)/(20*pi))  -sqrt(2)/(12*pi) 0 0 0 0;
    -1/6*pi  (sqrt(2)/12*pi)-(sqrt(2)/20*pi)  (sqrt(2)/4*pi)  sqrt(2)/12*pi  0 0 0 0;
-1/10*pi   -sqrt(2)/12*pi   sqrt(2)/12*pi   sqrt(2)/4*pi  0 0 0 0;
0 0 0 0  sqrt(2)/2*pi   1/6*pi  -9/(2*pi)  -1/10*pi;0 0 0 0 1/6*pi
 (sqrt(2)/(2*pi))-9*sqrt(2)/(4*pi)   (sqrt(2)/(12*pi))-sqrt(2)/20*pi   -9*sqrt(2)/4*pi;
    0 0 0 0  -9/2*pi   (sqrt(2)/12*pi)-sqrt(2)/(20*pi)   (sqrt(2)/2*pi)  (sqrt(2)/12*pi);
0 0 0 0 -1/10*pi   -9*sqrt(2)/4*pi   (sqrt(2)/12*pi)  (sqrt(2)/2*pi)];
b1= [sqrt(2)/pi  0  -2/3*pi  0;0  sqrt(2)/pi-sqrt(2)/3*pi  1/pi  -sqrt(2)/pi;
-2/3*pi  0  sqrt(2)/pi  0 ; 0 sqrt(2)/3*pi  0  sqrt(2)/pi];
o1= zeros (4);
 B1= [b1 o1; o1 b1];
d= tranP*A1;
I1= eye (8);
o= zeros (1, 8);
```

```
f=d-I1;
d1=tranP*B1;
c= [1.1284  1.5958 1.5958  1.5958   -1.1284  1.5958  -1.5958  -1.5958];% continuity matrix
F= [f  d1; c  o]; % A Matrix in linear equality constraints F*z = h
h=[(-1)*sqrt(pi)/2 0 0 0 (-1)*sqrt(pi)/2 0 0 0 0]'; %vector for linear equality constraints
a = zeros(1,16);    %Vector in linear term a'*x
Fineq = [ ];     % Matrix in linear inequality constraints F*z = h
hineq= [ ];     %Vector in linear inequality constraints F*z = h
lb = [ ];   %Vector of lower bounds
ub = [ ];   %Vector of lower bounds
x0 = 1;   %Initial point of x
opts=optimoptions('quadprog','Algorithm','interior-point-convex','Display','off'); % minimizes
with the default optimization parameters replaced by values in options, an argument created
with the optimoptions function.
[z ,J]=quadprog(H, a, Fineq, hineq, F, h, lb, ub, x0, opts);
display(z)
display(J)
a10=z(1);a11=z(2);a12=z(3);a13=z(4);a20=z(5);a21=z(6);a22=z(7);a23=z(8);
b10=z(9);b11=z(10);b12=z(11);b13=z(12);b20=z(13);b21=z(14);b22=z(15);b23=z(16);
X= a10*C10+a11*C11+a12*C12+a13*C13+a20*C10+a21*C21+a22*C22+a23*C23
U= b10*C10+b11*C11+b12*C12+b13*C13+b20*C10+b21*C21+b22*C22+b23*C23
```

Approximations of state and control variable using Legendre and Chebyshev scaling functions

Table 4. Approximations of state and control variable using Legendre and Chebyshev scaling functions of example 5.1.

Time (t)	Order poly	x(t)		u(t)	
		LSF	CSF	LSF	CSF
0.15	m=1	1.9653	1.4847	-0.7327	-0.6680
	m=2	1.5957	1.5947	-0.4479	-0.3556
	m=3	1.6182	1.6453	-0.6198	-3.8619
0.3	m=1	1.6333	1.2684	-0.6162	-0.5796
	m=2	1.3152	1.3144	-0.4220	-0.3420
	m=3	1.3238	1.3306	-0.4688	-1.8163
0.45	m=1	1.3013	1.0521	-0.4996	-0.4913
	m=2	1.0842	1.0836	-0.3227	-0.2789
	m=3	1.0855	1.0773	-0.3439	-0.6590
0.6	m=1	0.9693	0.8358	-0.3831	-0.4029
	m=2	0.9028	0.9022	-0.1500	-0.1664
	m=3	0.8935	0.8705	-0.2400	0.2847
0.75	m=1	0.6373	0.6195	-0.2665	-0.3145
	m=2	0.7709	0.7701	0.0962	-0.0045
	m=3	0.7383	0.6952	-0.1518	1.6894
0.9	m=1	0.3053	0.4033	-0.1500	-0.1672
	m=2	0.6885	0.6875	0.4157	0.2069
	m=3	0.6104	0.5365	-0.0744	4.2298

Table 5. Approximations of state and control variable using Legendre and Chebyshev scaling functions of example 5.2.

Time (t)	Order poly	x(t)		u(t)	
		LSF	CSF	LSF	CSF
0.15	m=1	1.4531	1.5532	-2.6802	-2.1678
	m=2	1.6671	1.6662	-2.4147	-2.4922
	m=3	1.6652	1.7678	-2.6770	-14.3976
0.3	m=1	1.3364	1.4977	-2.0592	-1.7808
	m=2	1.4404	1.4395	-1.9165	-1.9584
	m=3	1.4399	1.4689	-2.0376	-6.8221
0.45	m=1	1.2197	1.4087	-1.5405	-1.3938
	m=2	1.2820	1.2811	-1.4437	-1.4589
	m=3	1.2814	1.2833	-1.4973	-2.4359
0.6	m=1	1.1030	1.3720	-1.0218	-1.0068
	m=2	1.1919	1.1909	-0.9961	-0.9936
	m=3	1.1837	1.2033	-1.0335	1.8781
0.75	m=1	0.9863	1.3198	-0.5032	-0.6198
	m=2	1.1701	1.1689	-0.5738	-0.5626
	m=3	1.1410	1.2216	-0.6202	9.2372
0.9	m=1	0.8696	1.2309	0.0155	-0.2328
	m=2	1.1166	1.0940	-0.1768	-0.1658
	m=3	1.0475	1.1306	-0.2314	22.7585

Table 6. Approximations of state and control variable using Legendre and Chebyshev scaling Functions of example 5.3.

Time (t)	Order poly	x(t)		u(t)	
		LSF	CSF	LSF	CSF
0.15	m=1	1.2784	1.6492	-3.0390	-1.2336
	m=2	1.8931	2.6095	-2.7084	-6.9679
	m=3	1.5152	0.4999	-7.5219	20.8443
0.3	m=1	1.2271	1.5683	-2.1874	-1.0441
	m=2	1.6627	1.1326	-1.8851	-4.7175
	m=3	1.4647	0.6109	-3.4344	7.9199
0.45	m=1	1.1758	1.4874	-1.3359	-0.8545
	m=2	1.5245	0.9572	-1.3470	-2.6952
	m=3	1.4453	0.5353	-1.4903	2.4527
0.6	m=1	1.1245	1.4065	-0.4843	-0.6649
	m=2	1.4785	2.0833	-1.0941	-0.9011
	m=3	1.7409	0.1791	-1.5326	-0.1960
0.75	m=1	1.0732	1.3256	0.3672	-0.4753
	m=2	1.5247	4.5110	-1.1264	0.6648
	m=3	2.6350	-0.5514	-3.4040	-4.6649
0.9	m=1	1.0219	1.2447	1.2187	-0.2857
	m=2	1.6632	8.2402	-1.4438	2.0025
	m=3	4.4114	-1.7500	-6.9474	-15.5926