SOLVING MULTI-OBJECTIVE GEOMETRIC PROGRAMMING PROBLEMS WITH WEIGHTED SUM, FUZZY GEOMETRIC PROGRAMMING, ε −CONSTRAINED AND NEUROSOPHIC GEOEMETRIC PROGRAMMING METHODS

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DEDICATION

I dedicate this manuscript to my wife Simegnat Ayele and all my families

STATEMENT OF THE AUTHOR

By my signature below, I declare that this Project is my own work. I have followed all ethical and technical principles of scholarship in the preparation, and compilation of this Project. Any scholarly matter that is included in the Project has been given recognition through citation. This project is submitted in partial fulfillment of the requirements for MSc degree in Mathematics with specialization in optimization at Haramaya University. The Project is deposited in the Haramaya University Library and is made available to borrowers under the rules of the Library. I solemnly declare that this Project has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate. Brief quotations from this Project may be made without special permission provided that accurate and complete acknowledgement of the source is made. Requests for permission for extended quotations from or reproduction of this Project in whole or in part may be granted by the Head of the Department when in his or her judgment the proper use of the material is inthe interest of scholarship. In all other instances, however, permission must be obtained fromthe author of the Project.

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LIST OF ABBREVIATIONS

DD	Degrees of Difficulty
FGPP	Fuzzy Geometric Programming Problem
GP	Geometric Programming
MOGPP	Multi-Objective Geometric Programming Problem
MOO	Multi-Objective Optimization
NGPP	Neurosophic Geometric Programming Problem
OR	Operations Research
SOGPP	Single Objective Geometric Programming Problems
WSM	Weighted Sum Method

BIOGRAPHICAL SKETCH

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ABSTRUCT

In this project, we discussed multi objective geometric programming problems by covering some basic concepts as well as different solution methods. Methods like weighted sum, ε-constraint, fuzzy geoemetric programming and Neutrosophic geometric programming methods were applied for solving multi-objective geometric programming optimization problems. First, a multi objective geometric programming problems is transformed into a single-objective geometric programming optimization problem by using weighted sum, ε-constraint, fuzzy and neutrosophic geometric programming methods. Then the transformed single-objective geometric programming optimization problem was solved by arthemetic geometric inequality or geometric programming method. Illustrative examples were presented to show the effectiveness of the proposed methods and the results so obtained by weighted sum method has been compared with fuzzy and neutrosophic geometric programming methods. From the result we concluded that FGP method obtained the sum of two objective functions are 65.6, NGP method obtained 65.35 and weighted sum 64.807 which is the sum of two objective functions. This implies that weighted sum method is the most appropriate to solve MOGPP than other listed methods because the result obtained is too minimum relatively to others and neutrosophic method is the second best method according to the result obtained. We can recommend that all problems solved by weighted sum method can not solved by other problems but problems solved by FGP, NGP and epsilon constrained can also be solved by weighted sum method. This is due to the degree of difficulty being negative. so all transformed MOGPP to SOGPP can not be solved by GP method if degree of difficulty is negative it requires further investigation and also as the degree of difficulty increases it is difficult to solve.

Key words: Multi-Objective Geometric Programming**, F**uzzy Geoemetric Programming, Weighted Sum Method, E-Constraint And Neutrosophic Geometric Programming Method

1. INTRODUCTION

1.1. Background of the Study

Optimization is a mathematical problem with many real world applications. The goal is to determine minimizers or maximizers of a multivariable real function, under a restricted domain. The aim optimization is to maximize or minimize a function (e.g. maximizing profit or minimizing environmental impact) subject to a set of constraints. However, in many situations, decision makers find themselves wanting to optimize several different objective functions at the same time. Optimization problems are real world problems we encounter in many areas such as mathematics, engineering, science, business and economics. In these problems, we find the optimal, or most efficient, way of using limited resources to achieve the objective of the situation (Ojha and Rashmi, 2014).

Geometric Programming (GP) is used to solve a class of non-linear programming problems to minimize an objective function subject to certain constrains which are in the form of posynomials that has positive coefficients and any types of exponents. Duffin(1967), put a foundation stone to solve wide range of engineering design problems by developing basic theories of Geometric Programming. Geometric Programming also is a powerful optimization technique developed to solve non-linear optimization programming problems especially found in engineering design and manufacturing. GP problems have wide range of application in production planning, location, distribution, risk managements, chemical process designs and other engineering design situations. The solution procedure for a Geometric programming problem may be categorized as of two types. It is either primal based algorithms that directly solve the non-linear primal problem, or dual based algorithms that solve the equivalent linear constraint dual program (Ersoy*et al.*, 2017).

Nowaday, GP techniques have been used extensively to solve various engineering design problems which are in the form of multi-objective functions. This indicates that multi-objective optimization problems do not have a single solution that could optimize all objective functions simultaneously. However, the decision makers are always in search of a most compromise solution that could optimize all objective functions.

One of the important characteristic of GP is that a problem with highly non-linear constraints can be converted to a problem with linear constraints using its duality theorem, so that the problem will be easy to solve. It has certain advantages over the other optimization methods. Here, the advantage is that it is usually much simpler to work with the dual than the primal one. GP is such a powerful optimization technique whose elegant theoretical concept has led a number of researchers to develop its interesting application in various fields (Sahidul, 2010).

Several methods have been proposed to solve various non-linear programming problems subject to linear and non-linear constraints. GP technique is an excellent method when decision variable interact in a non-linear optimization problem. Most of these GP applications are posynomial type with zero or few degrees of difficulty. Non-linear optimization problems whose parameters, except for exponents, are all positive are called posynomial problems, where Non-linear optimization problems with some negative parameters are referred to as signomial problems. The degree of difficulty (DD) is defined as the number of terms minus the number of variables minus one, and is equal to the dimension of the dual problem. When the degree of difficulty is zero, the problem can be solved analytically. For such posynomial problems, GP techniques find global optimal solutions. If the degree of difficulty is positive, then the dual feasible region must be searched to maximize the dual objective, while if the degree of difficulty is negative, the dual constraints may be inconsistent. Degree of difficulty is an important concept in the classical GP theory. The dual problem is often infeasible when the degree of difficulty is negative (Abbas and Huda, 2012).

This project presented the basic concepts and principles of multiple-objective geometric programming problems, and already developed a numerical procedure to solve multi-objective Geometric Programming Problems having zero and positive types of degree of difficulties using weighted sum, $\varepsilon -$constraint, fuzzy and Neutrosophic methods to obtain the optimal compromise solution. If the degree of difficulty of MOGPP is negative, the dual constraints could be inconsistent. Hence in this case the dual problem presents a system of linear equations, and the number of these linear equations is greater than the number of dual variables. So the dual problem possesses no feasible solution. In general there is no solution vector for the dual variables in this case (Abbas and Huda, 2012).

A mathematical MOGP optimization problem has the form:

Find $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)^T$ so as to

$$\left.\begin{array}{c}\min f_k(x) = \sum_{t=1}^{T_k} C_{tk} \prod_{j=1}^{n} x_j^{a_{tj}^k}, k = 1,2,\ldots,p \\ \text{subject to } g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1, i = 1,2,\ldots,m \\ x_j > 0, j = 1,2,\ldots,n\end{array}\right\} \quad (1.1)$$

where:$C_{tk} > 0$,$C_{it} > 0$; $a_{tj}^k$and a_{itj} are any real number for all i, j, k, t and $T_k =$ Number of terms present in k^{th} the objective function $f_k(x)$; T_i= number of terms present in the i^{th} constraint. In the above multi-objective Geometric programming problem, there are p number of minimization type objective functions, m number of inequality type constraints and n number of strictly positive decision variables. The multi-objective Geometric programming problem defined in (1.1) is considered as a Vector minimization problem. It is assumed that the problem has an optimal solution (Ota and Ojha, 2015).

We used different methods to find optimal compromise solution of certain multi-objective geometric programming problem of the form (1.1) such as weighting, epsilon constrained, fuzzy geometric programming and neutrosophic methods. Fuzzy programming Problem due to Zimmermann (1990), based onthe concept given by Bellman and Zadeh (1965), has been successfully applied to solve various types of multi-objective decision making problems such as engineering design and maintenance, production planning and control, transportation, water resource management, managerial decision making and scheduling problems. A fuzzy set is associated with its membership function which is defined from its elements to the interval $[0,1]$ which plays an important role in solving multi-objective decision making problems. As there are several types of fuzzy membership functions,a suitable membership function is to be selected to solve the real world multi-objective mathematical programming problems.

The weighting method is the simplest multi-objective optimization which has been widely applied to find the optimal solutions of multi-objective function within the convex objective space. If $f_1(x), f_2(x), f_3(x), f_4(x), \ldots, f_K(x)$ are k objective functions for any vector $\boldsymbol{x} = (x_1, x_2, x_3, \ldots, x_n)^T$ then we can define weighting method for their optimal solution as: let $\{\mathrm{w}: \mathrm{w} \in \mathbb{R}^n, \mathrm{w_j} > 0, \sum_{j=1}^{n} \mathrm{w_j} = 1\}$ be the set of $\text{non} - \text{negative wieghts}$. Weighted function for the multiple-objectives optimization functions defined above in (1.1) can be written as $\min_{x=X} \sum_{j=1}^{k} w_j f_j(x)$ it must be made clear, however, that if the objective space of the original problem is non-convex, then the weighting method may not be capable of generating the efficient solutions on the non-convex problems. It must also be noted that the optimal solution of a weighting problem should not be used as the best optimal solution. Based on the importance of the k number of objective functions defined in (1.1) the weights $w_1, w_2, w_3, \dots, w_k$ are assigned to define a new minimum type objective function.

In this project we also used, ε-constraint method which used to solve multi-objective Geometric programming problems for searching an optimal compromise solutions. To find the suitable compromise solution for multi-objective Geometric programming problems, a brief solution procedure using ε-constraint method was presented (Ojha and Rashmi, 2014).

The ε-constraint method is used to solve a class of multi-objective Geometric programming problems. Using ε-constraint method, we can optimize one of the objective function at a time where other objectives are kept in the constraint. This method is found more suitable than other generating methods used for obtaining optimal solutions. After obtaining lower and upper bounds of each objective function with the given constraints, we have generated a set of optimal solution. The results obtained by epsilon constrained method was compared with its corresponding solution obtained by fuzzy programming method and weighted sum method. Biswal (1992), in their recent paper have studied the multi-objective optimization problems by solving a series of single objective sub-problems, where all but one objective are transformed into constraints. In this project the solution of dual variable not only gives the primal optimal solution but also provides a relationship between maximum and minimum value of the objective function. The same results were obtained when converting multi-objective Geometric programming problem to a single objective problem using the mentioned methods such as Weighted sum method, fuzzy method, $\varepsilon -$constrained and neutrosophic method for verification.

1.2. Statement of the Problem

Multi-objective posynomial geometric programming optimization is a class of problems with equality and inequality constraints have solutions that can be evaluated along two or more objectives functions. These types of problems differ from standard optimization problems in that the end result is not a single solution.When we solve the problem of the form (1.1), we consider degree of difficulty which showed us the dimension of the dual problem. When the degree of difficulty is greater than zero, it is difficult to change the problem from the primal to the dual form.There are some methods of solving non-linear programming problems but still the best is yet to be achieved. In this project, we used FGP, NGP, epsilon constained and WSM in solving MOGGP to see whether a better solution can be obtained when compared the sulution obtained by listed methods.

This project analyzed solution methods of multi-objective geometric programming problems. A multi-objective geometric programming problem can be stated as:
Find $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)^T$ so as to

$$\left.\begin{aligned} \min f_k(x) &= \sum_{t=1}^{T_k} C_{tk} \prod_{j=1}^{n} x_j^{a_{tj}^k}, k = 1,2,\dots,p \\ \text{subject to} g_i(x) &= \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1, i = 1,2,\dots,m \\ & x_j > 0, j = 1,2,\dots,n \end{aligned}\right\} \quad (1.1)$$

$C_{tk} > 0, C_{it} > 0$; a_{tj}^k and a_{itj} are any real number for all i, j, k, t and $T_k =$ number of terms present in k^{th} the objective function $f_k(x)$; T_i= number of terms present in the i^{th} constraint.In the above multi-objective posynomial geometric program there are p number of minimization type objective function, m number of inequality type constraints and n number of strictly positive decision variables.

1.3. Objectives

The main objective of this project was to solve a multi-objective geometric programming problem of type (1.1) by using weighted sum, fuzzy geometric programming, ϵ −constrainedand neutrosophic geometric programming methods.

The study explored the following specific objectives.

- To solve multi-objective geometric programming problems using weighted sum, fuzz geometric programming, ϵ −constrained and neutrosophic geometric programming methods.
- To compare compromise solutions obtained by weighted sum, neutrosophic geometric programming and fuzz geometric programming methods which transforming the problem (1.1) to SOGPP.
- To applymulti-objective geometric programming problems in the real world and solve with weighted sum method.

2. REVIEW OFRELATED LITERATURE

2.1. Single Objective Geometric Programming Problems

Geometric Programming is an optimization technique developed for solving a class of non-linear optimization problems with some useful theoretical and computational properties. These optimization problems are not convex in their natural form, they can however, be transformed into convex optimization problems by a change of variables and a transformation of the objective and constraint functions. Fein (1961) in a seminar paper observed that some engineering design problems can be formulated as optimization of generalized polynomials and that if the number of terms exceed the number of variables by one, the optimal design can be found by solving a system of linear equations. Geometric programming (GP) was tied with convex optimization and Lagrange duality and extended to include more general formulations beyond polynomials.

Geometric programming addresses optimization program where the objective functions are sums of monomials. The sum of monomials with positive signs is called Posynomials. If some of the monomials enter the sum with negative sign, the collection is called Signomials. The term Geometric programming was adopted because of the crucial role that the arithmetic mean inequality played in the initial development. The early works by Zener et al. (1963) in geometric programming was for most part, concerned with minimizing posynomial functions subject to inequality constraint on such functions. Thus, the name posynomial programming might well have been chosen instead if geometric programming.

The Primal problem on the other hand, is usually considerably more non-linear than its associated dual problem. Geometric programming in standard form is apparently non convex optimization problem. It can be readily turned into a convex optimization problem by a logarithmic change of the variables and multiplicative constants. Zener (1963) used the result called Cauchy's arithmetic-geometric inequality to show that the arithmetic mean of a group terms always was greater than or equal to the geometric mean of the group. It is for this reason, that the name Geometric Programming was used to describe the class of non-linear optimization problems.

Geometric Programming problems have wide range of application in production planning, location, distribution, risk managements, chemical process designs and other engineer design situations. GP problem is an excellent method when decision variable interact in a non-linear, especially, in an exponential fashion. Most of these GP applications are posynomial type with zero or few degrees of difficulty. GP problem whose parameters, except for exponents, are all positive are called posynomial problems, whereas GP problems with some negative parameters are referred to as signomial problems(Ojha and Biswal, 2010).

The degree of difficulty in constrained posynomial geometric programming problems is defined as the number of terms minus the number of variables minus one, and is equal to the dimension of the dual problem. When the degree of difficulty is zero, the problem can be solved analytically. For such posynomial problems, GP techniques find global optimal solutions.If the degree of difficulty is positive, then the dual feasible region must be searched to maximize the dual objective,while if the degree of difficulty is negative, the dual constraints may be inconsistent (Beightler*et al*.,1979).

One of the remarkable properties of Geometric programming is that a problem with highly nonlinear constraints can be stated equivalently with a dual program. If a primal problem is in posynomial form then a global minimizing solution of the problem can be obtained by solving its corresponding dual maximization problem because the dual constraints are linear, and linearly constrained programs are generally easier to solve than ones with nonlinear constraints. GP problem has a dual impact in the area of integrated circuit design, manufacturing system design,project management, maximization of long run and short term profit, and generalized geometric programming problem with non-positive variables (Ojha and Biswal, 2010).

The arithmetic mean and geometric mean inequality (also known as the arithmetic–geometric inequality or Cauchy's inequality) is very useful in solving geometric programming problems. Using the inequality, the objective function is that the maximum of the dual function equals the minimum of theprimalfunction. The optimization by minimizing the primal or by maximizing the dual,whichever is easier. Also, the maximization of the dual function subject to the orthogonalityand normality conditions is a sufficient condition the primal function f, tobe a global minimum and we saw that geometric programming treats the problemof minimizing posynomials and maximizing product functions. The minimizationproblems are called primal programs and the maximization problems are called dual programs(Duffin et al., 1967).

2.2. Multi-Objective Geometric Programming Optimization Problem

The classical framework for optimization is to find the optimum value of objective functions with respect to the given constraints. All the conventional type optimization methods seek to find a single optimal solution based on a weighted sum of all objectives. If all objectives get better or worse together, then conventional approach can effectively find the optimal solution. In this case, a multi-objective optimization study should be performed which provides multiple solutions among the objectives (Biswal, 1992).

A solution of multi-objective optimization problem is considered to be more a concept than a definition. In multi-objective optimization problems, what is optimal in terms of one of the objectives is usually non-optimal for the remaining objectives. Consequently, there is no single optimal solution exist for a multi-objective optimization problem. Hence we have to search for a solution which is acceptable to the decision maker. The method of optimizing systematically and simultaneously a collection of objective function are called multi-objective optimization. Geometric Programming problems has been known and used in various fields since 1960. GPP as part of nonlinear optimization particular algorithms were used when trying to solve GPP. When there are multiple objectives in the GPP, the problem is defined as the Multi-Objective Geometric Programming Problem (Duffin*et al.,* 1961).

However, the decision makers search for most compromise solution for all objectives. First of all, the multiple objective functions transformed to a single objective by considering it as the linear combination of the multiple objectives along with suitable constants called weights. By changing the weights,the most compromise optimal solution will be arrived by using GP techniques (Verma,1990).

Ojha and Biswal (2010), stated that weighted sum method is the simplest method widely used to transform a set of objectives into a single objective by multiplying each objective with weights to find the optimal compromise solution of a multi-objective optimization problem within the convex objective space.If$f_1(x), f_2(x), f_3(x), f_4(x), \dots, f_k(x)$ are k objective functions for any vector$x = (x_1, x_2, x_3, \dots, x_n)^T$, then we can define weighting sum method is as follows:
Let $\mathrm{W} = \{\mathrm{w}: \mathrm{w} \in \mathbb{R}^{\mathrm{n}}, \mathrm{w}_{\mathrm{k}} > 0, \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{w}_{\mathrm{k}} = 1\}$ be the set of non-negative weights and by using weighting method the multi-objective function with constraints can be defined as:

$$\left.\begin{array}{c} Q(x) = \min_{x \in X} \sum_{k=1}^{p} w_k\, f_k(x) \\ \text{subject to } g_i(x) \leq 1, i = 1,2,\dots,m \\ x_j > 0, j = 1,2,\dots,n \end{array}\right\} \quad 1.2$$

It is necessary that the objective space of original problem (1.1) should be convex. If it is not convex, then weighting method is not capable of generating the efficient solutions on the non-convex optimization problem. It must be noted that the optimal solution of an optimization problem using weighting method should not be accepted as the best compromise solution if that do not reflect in decision makers mind. Based on importance of p number of objective functions defined in (1.1) the weights $w_1, w_2, w_3, w_4, \dots, w_p$ are assigned to define a new minimization type objective function $F(X)$which can be definedas:

$$\left.\begin{array}{c} \min_{x} F(x) = \sum_{k=1}^{p} w_k\, f_k(x) = \sum_{k=1}^{p} w_k \left(\sum_{t=1}^{T_0^k} C_{tk}^0 \prod_{j=1}^{n} x_j^{a_{0tj}^k} \right) \\ x_j > 0, j = 1,2,\dots,n \\ where \qquad \sum_{k=1}^{p} w_k = 1, k = 1,2,\dots,p \end{array}\right\} \quad 1.3$$

Now after this formulation we use the above equation (1.3) which is transformed into dual program in order to solve the problem either primal based algorithms that directly solve the non-linear primal problem, or dual based algorithms that solve the equivalent linear constraint dual program.

Haimes *et al.* (1971) a method which overcomes some of the convexity problems of the weighted sum technique is known as ϵ-constraint method. This method involves minimizing a primary objective and expressing the other objectives in the form of inequality constraints. The ϵ-constrain method is used for generating optimal solutions for the multi-objective optimization problem. This method generates the optimal solutions of multi-objective optimization problems by considering one objective function at a time as primary one and converting the remaining objective functions as constraints. In other word it minimizes one objective function and simultaneously maintains the maximum acceptability level for other objective function.

For any non-linear multi-objective optimization problem, the solution obtained by ϵ-constraint method yields optimal solution. Optimal solution can be obtained, either if the solution is unique or if the optimizations are done for all the objectives before reporting the solution. The proposed method will design to deals with the real valued problems which are likely to have a systematic variation of ϵ_j will yields a set of solution. However, the determination of the minimum level and assumption about the form of preference in finding the preferred decisions are often questionable in real world problems. The epsilon constraint method is also applicable to a non-convex optimization problem (Verma,1990).

The ϵ-constraint method involves minimizing a primary objective and expressing the other objectives in the form of inequality constraints. The ϵ-constraint method was proposed by Haimes et al.(1979), for generating optimal solutions for the multi-objective optimization problem. This method generates optimal solutions of multi-objective optimization problems by considering one objective function at a time as primary one and converting the remaining objective functions as constraints. In other word it minimizes one objective function and simultaneously maintains the maximum acceptability level for other objective function.

2.3. Degree of Difficulty of Geometric Programming Problems

Degree of difficulty is an important concept in the classical Geometric Programming theory. The dual problem is often infeasible when the degree of difficulty is negative. This will present the basic concepts and principles of multiple-objective geometric programming problems and solution procedures to solve multi-objective Geometric Programming Problems having a negative degree of difficulty using the three methods is described in the introduction above. Geometric Programming Problems are smooth non-linear programs in which the objective and each constraint function is a posynomials which is linear combination of terms with each term product of variables raised to real powers and each constraint function (Abbas and Huda, 2012).

The decision variables are restricted to be positive, to ensure that terms involving variables raised to fractional powers are defined. If all the linear combination coefficients are positive, the functions are called posynomials. Most of GPP applications are posynomials type with zero or a few degree of difficulty. The degree of difficulty of a GP problem in equation (1.1) can be computed as:

$$DD(\text{Degree of Difficulty}) = \sum_{k=1}^{p} T_0^k + \sum_{i=1}^{m} T_i - n - 1$$

This DD determines the types of solution that the given multi-objective geometric programming problems have and we can see in three ways. 1) If the degree of difficulty is zero, then the problem (1.1) has unique solution, 2) If the degree of difficulty is positive, then the problem has much solution and 3) If the degree of difficulty is negative, then the above methods are not capable to solve suchtypes of problems which is inconsistent (Abbas and Huda, 2012).

3. METHODOLOGY

In this project, we used four methods to solve multi objective geometric programming problems. These are weighted sum, ϵ-constraint, fuzzy geometric programming and neutrosophic geoemetric programming methods sused to solve multi-objective Geometric programming problems for searching a compromise solution.

The basic concept and classical procedures of multi-objective optimization problems discussed. Illustrative examples which have positive degree of difficulty was presented to demonstrate the correctness of proposed procedures. Sources were organized from different directions such as in libraries, you tube, journals and books used to gather information about multi-objective Geometric programming optimization problems.

Important concepts, definitions, examples and graphs were considered to make ideas clear about multi-objective Geometric Programming problems. All listed methods followed different procedures to transform multi-objective geometric programming problems into single objective geometric programming problems. Then we used the same procedures to obtain the optimal compromise solutions of the desired multi objective geometric programming problems (1.1).

4. PRELIMINARIES

4.1. Basic Geometric programming

4.1.1.Monomial

Let $x_1, \dots, x_n$ denote n real positive variables, and $x = (x_1, \dots, x_n)$ a vector with components x_i A real valued function $f$of x, with the form:

$$f(x) = cx_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} \dots x_n^{a_n}$$

where $c > 0$ and a_i are any real number is called a monomial functions, or more informally, a monomial of variables $x_1, \dots, x_n$ and constant c as coeffiecent of the monomial and we refer the constants $a_1, a_2, a_3 \dots, a_n$ as the exponents of the monomial. Any positive constant is a monomial, as is any variable. Monomials are closed under multiplication and division: if $f\ and\ g$ are both monomials then so are $fg\ and \frac{f}{g}$ this includes scaling by any positive constant. A monomial raised to any power is also a monomial.

4.1.2. Posynomial

The sum of one or more monomials of the function the form:

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}$$

Where $c_k > 0,$ is called a posynomial function or more simply a posynomial with k terems in the variable $x_1, \dots, x_n$ is the combination of positive polynomials. Any monomial is also a posynomial. Posynomials are closed under addition, multiplication, and positive scaling. Posynomials can be divided by monomials (with the result also a posynomial. If f is a posynomial and g is a monomial, then $\frac{f}{g}$ is a posynomial.

4.1.3. Geometric Programming

Geoemetric programming was developed by Duffin, Zener and Peterson for solving the class of optimization problems involving special functions called posynomials. It is very efficient technique for certain highly non-linear and non-convex problems. This technique is based on the AM-GM inequality and therefore called geoemeteric programming. Geometric programming always transforms the primal problem of minimizing a posynomial subject to posynomial constraints to a dual problem of maximizing a function of the weights on each constraint. Posynomial functions can be defined as polynomials in several variables with positive coefficients in all terms and the power to which the variables are raised can be any real number (Shafiullah *et al.*, 2015).

4.1.4.Arthematic mean geometric mean inequalities

Let $U_1, U_2, U_3, U_4, \dots, U_n$ be a non- negative numbers. Suppose $\delta_1, \delta_2, \delta_3, \delta_4, \dots, \delta_n$ be such that $\delta_i > 0$ for all $i = 1,2,3, \dots, n$ and $\delta_1 + \delta_2 + \delta_3 + \delta_4 + \cdots + \delta_n = 1$. Now Arthimetic mean $\geq$ Geoemeteric mean. We use the above information and can be written as:

$$\frac{\delta_1 U_1 + \delta_2 U_2 + \delta_3 U_3 + \delta_4 U_4 + \delta_5 U_5 + \cdots + \delta_n U_n}{\delta_1, \delta_2, \delta_3, \delta_4, \dots, \delta_n} \geq \ {U_1}^{\delta_1} {U_2}^{\delta_2} {U_3}^{\delta_3} {U_4}^{\delta_4} {U_n}^{\delta_n}$$

$$\delta_1 u_1 + \delta_2 u_2 + \delta_3 u_3 + \delta_4 u_4 + \delta_5 u_5 + \cdots + \delta_n u_n \ \geq \ {u_1}^{\delta_1} {u_2}^{\delta_2} {u_3}^{\delta_3} {u_4}^{\delta_4} {u_n}^{\delta_n}$$

$$\sum_{i=1}^{n} \delta_i u_i \geq \prod_{i=1}^{n} {u_i}^{\delta_i}$$

Let $\delta_i = \frac{1}{n}$ for all i. Then substitute above equation and we get the following inequality.

$$\sum_{i=1}^{n} \left(\frac{u_i}{n}\right) \geq \prod_{i=1}^{n} {u_i}^{1/n}$$

This implies that

$$\frac{u_1 + u_2 + u_3 + u_4 + u_5 + \cdots + u_n}{n} \geq (u_1 u_2 u_3 \dots . u_n)^{1/n}$$

Let $U_i = \delta_i u_i$ then substitute above equation and we get that:

$$\sum_{i=1}^{n} (\delta_i u_i) = \sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i}$$

Therefore this helps us to solve non-linear geometric programming problems.

$$\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i}$$

The equality condition holds true when

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \frac{U_3}{\delta_3} = \cdots = \frac{U_n}{\delta_n} = k$$

Then we have that $U_1 = \delta_1 k, U_2 = \delta_2 k, U_3 = \delta_3 k, \dots, U_n = \delta_n k$

$$\sum_{i=1}^{n} U_i = U_1 + U_2 + U_3 + \cdots + U_n$$

$$= \delta_1 k + \delta_2 k + \delta_3 k + \cdots + \delta_n k$$

$$= \boldsymbol{k}(\delta_1 + \delta_2 + \delta_3 + \cdots + \delta_n)$$

$$= \boldsymbol{k} * \mathbf{1}$$

$$= \boldsymbol{k}$$

$$\prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i} = \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \left(\frac{U_3}{\delta_3}\right)^{\delta_3} \dots \left(\frac{U_n}{\delta_n}\right)^{\delta_n}$$

$$= \left(\frac{\delta_1 k}{\delta_1}\right)^{\delta_1} \left(\frac{\delta_2 k}{\delta_2}\right)^{\delta_2} \left(\frac{\delta_3 k}{\delta_3}\right)^{\delta_3} \dots \left(\frac{\delta_n k}{\delta_n}\right)^{\delta_n}$$

$$= (k)^{\delta_1} (k)^{\delta_2} (k)^{\delta_3} \dots (k)^{\delta_n}$$

$$= (k)^{\delta_1 + \delta_2 + \delta_3 + \cdots + \delta_n}$$

$$= \boldsymbol{k}$$

Therefore from two cases we can concluded that equality condition holds true and helps us to find the primal variables.

4.1.5. Unconstrained posynomial geometric programming optimization

Consider the following problem (GP):

$$\left.\begin{array}{c} \text{Min } f(x) = \sum_{j=1}^{p} c_j u_j\,(x), c_j > 0 \text{ and } u_j(x) \\ u_j(x) = \prod_{i=1}^{m} (x_i)^{a_{ij}} = (x_1)^{a_{1j}} (x_2)^{a_{2j}} (x_3)^{a_{3j}} (x_4)^{a_{4j}} (x_5)^{a_{5j}} \dots (x_n)^{a_{mj}} \\ x_i > 0 \end{array}\right\} \quad 1.2$$

Therefore we can rewrite it as:

$$\text{Min } f(x) = \sum_{j=1}^{p} c_j \prod_{i=1}^{m} (x_i)^{a_{ij}}$$

Where $a_{ij}, 1 \leq i \leq m \; and \; 1 \leq j \leq n$ be real numbers and $x_i > 0, i = 1,2,3, \dots, m$. Then we call $u_j(x)$ a posynomial optimization.

Example 4.1.5: solve the following unconstrained posynomial geometric programming problems.

$$\text{Min} f(x) = x_1 + x_2 + \frac{1}{x_1 x_2}$$

$$\text{where } x_1, x_2 > 0$$

$$f(x) = x_1 + x_2 + \frac{1}{x_1 x_2}$$

$$= u_1 + u_2 + u_3$$

$$\geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \left(\frac{U_3}{\delta_3}\right)^{\delta_3}$$

$$= \left(\frac{x_1}{\delta_1}\right)^{\delta_1} \left(\frac{x_2}{\delta_2}\right)^{\delta_2} \left(\frac{\frac{1}{x_1 x_2}}{\delta_3}\right)^{\delta_3}$$

$$= {x_1}^{\delta_1 - \delta_3} {x_1}^{\delta_2 - \delta_3} \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{1}{\delta_3}\right)^{\delta_3}$$

$$= \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{1}{\delta_3}\right)^{\delta_3}$$

$$s.t. \begin{cases} \delta_1 - \delta_3 = 0 \\ \delta_2 - \delta_3 = 0 \\ \delta_1 + \delta_2 + \delta_3 = 1 \\ \delta_i > 0 \end{cases}$$

From above we can understand that the maximum values of the right side is equal to the lower bounds of the left side. In order to find the maximum values of the the right side of the equation, we must make it free from the variable (primal variable) because the values of the variable are not known. $\delta_1 = \frac{5}{9}, \delta_2 = \frac{4}{9}, \delta_3 = \frac{5}{9} \; and \; \delta_4 = \frac{2}{9}$, then we substitute in above inequality equation of the right side as follows:

$$\min f(x) \geq {x_1}^{\delta_1 - \delta_3} {x_1}^{\delta_2 - \delta_3} \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{1}{\delta_3}\right)^{\delta_3}$$

$$= {x_1}^{\frac{1}{3}-\frac{1}{3}} {x_1}^{\frac{1}{3}-\frac{1}{3}} \left(\frac{1}{\frac{1}{3}}\right)^{\frac{1}{3}} \left(\frac{1}{\frac{1}{3}}\right)^{\frac{1}{3}} \left(\frac{1}{\frac{1}{3}}\right)^{\frac{1}{3}}$$

$$= (3)^{\frac{1}{3}}(3)^{\frac{1}{3}}(3)^{\frac{1}{3}}$$

$$= 3$$

Therefore $\min f(x) \geq 3$ which is the minimum values of the unconstrained function given above. We can also find the primal variables using the equality conditions of the AM-GM inequality as follows.

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \frac{U_3}{\delta_3}$$

$$\frac{x_1}{\delta_1} = \frac{x_2}{\delta_2} = \frac{1}{x_1 x_2 \delta_3}$$

This implies that $x_1 = x_2 = 1$ are the optimal solutions of the primal variables.

4.1.6.Geometric programming with equality constraints optimization problems

Consider the case of minimizing an objective function which is the sum of posynomials subject to the equality constraints. That is:

$$\left.\begin{aligned} \text{Min} Z &= f(x) \\ g_i(x) &= \sum_{r=1}^{p_i} C_{ir} U_{ir}(x) \leq 1, i = 1,2,3 \ldots, n \end{aligned}\right\} \quad 1.3$$

Where p_i denotes the number of terms in the i$^{\text{th}}$ constraint and $U_{ir}(x) = \prod_{i=1}^{n} (x_i)^{a_{jir}}$

Now we can apply the AM-GM inequality condition for constrained problemas follows:

$$\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i}, \delta_1 + \delta_2 + \delta_3 + \cdots .. + \delta_n = 1 \; and \; \delta_i > 0$$

$$\delta_1 + \delta_2 + \delta_3 + \cdots .. + \delta_n = \boldsymbol{\lambda}$$

This implies that $\frac{\delta_1}{\boldsymbol{\lambda}} + \frac{\delta_2}{\boldsymbol{\lambda}} + \frac{\delta_3}{\boldsymbol{\lambda}} + \cdots + \frac{\delta_n}{\boldsymbol{\lambda}} = 1$, each term again consider as new δ_i and their sum is one.

$$\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} \left(\frac{U_i}{\delta_i}\right)^{\delta_i} \sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} \left(\frac{U_i}{\frac{\delta_i}{\lambda}}\right)^{\frac{\delta_i}{\lambda}}$$

$$= \left(\frac{U_1}{\frac{\delta_1}{\lambda}}\right)^{\frac{\delta_1}{\lambda}} \left(\frac{U_2}{\frac{\delta_2}{\lambda}}\right)^{\frac{\delta_2}{\lambda}} \left(\frac{U_3}{\frac{\delta_3}{\lambda}}\right)^{\frac{\delta_3}{\lambda}} \left(\frac{U_4}{\frac{\delta_4}{\lambda}}\right)^{\frac{\delta_4}{\lambda}} \cdots \left(\frac{U_n}{\frac{\delta_n}{\lambda}}\right)^{\frac{\delta_n}{\lambda}}$$

$$= \left(\frac{\boldsymbol{\lambda} U_1}{\delta_1}\right)^{\frac{\delta_1}{\lambda}} \left(\frac{\boldsymbol{\lambda} U_2}{\delta_2}\right)^{\frac{\delta_2}{\lambda}} \left(\frac{\boldsymbol{\lambda} U_3}{\delta_3}\right)^{\frac{\delta_3}{\lambda}} \left(\frac{\boldsymbol{\lambda} U_4}{\delta_4}\right)^{\frac{\delta_4}{\lambda}} \cdots \left(\frac{\boldsymbol{\lambda} U_n}{\delta_n}\right)^{\frac{\delta_n}{\lambda}}$$

$$= (\boldsymbol{\lambda})^{\frac{\delta_1}{\lambda}} (\boldsymbol{\lambda})^{\frac{\delta_2}{\lambda}} (\boldsymbol{\lambda})^{\frac{\delta_3}{\lambda}} (\boldsymbol{\lambda})^{\frac{\delta_4}{\lambda}} \ldots (\boldsymbol{\lambda})^{\frac{\delta_n}{\lambda}} \left(\frac{U_1}{\delta_1}\right)^{\frac{\delta_1}{\lambda}} \left(\frac{U_2}{\delta_2}\right)^{\frac{\delta_2}{\lambda}} \left(\frac{U_3}{\delta_3}\right)^{\frac{\delta_3}{\lambda}} \left(\frac{U_4}{\delta_4}\right)^{\frac{\delta_4}{\lambda}} \cdots \left(\frac{U_n}{\delta_n}\right)^{\frac{\delta_n}{\lambda}}$$

$$= \boldsymbol{\lambda}^{\frac{\delta_1}{\lambda}+\frac{\delta_2}{\lambda}+\frac{\delta_3}{\lambda}+\cdots+\frac{\delta_n}{\lambda}} \left(\frac{U_1}{\delta_1}\right)^{\frac{\delta_1}{\lambda}} \left(\frac{U_2}{\delta_2}\right)^{\frac{\delta_2}{\lambda}} \left(\frac{U_3}{\delta_3}\right)^{\frac{\delta_3}{\lambda}} \left(\frac{U_4}{\delta_4}\right)^{\frac{\delta_4}{\lambda}} \cdots \left(\frac{U_n}{\delta_n}\right)^{\frac{\delta_n}{\lambda}}$$

$$= \boldsymbol{\lambda} \left(\frac{U_1}{\delta_1}\right)^{\frac{\delta_1}{\lambda}} \left(\frac{U_2}{\delta_2}\right)^{\frac{\delta_2}{\lambda}} \left(\frac{U_3}{\delta_3}\right)^{\frac{\delta_3}{\lambda}} \left(\frac{U_4}{\delta_4}\right)^{\frac{\delta_4}{\lambda}} \cdots \left(\frac{U_n}{\delta_n}\right)^{\frac{\delta_n}{\lambda}}$$

$$= \boldsymbol{\lambda} \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \left(\frac{U_i}{\delta_i}\right)^{\frac{\delta_i}{\lambda}}$$

Then both side multiply power of λ and we get

$$\left(\sum_{i=1}^{n} U_i\right)^{\boldsymbol{\lambda}} \geq \left(\boldsymbol{\lambda} \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \left(\frac{U_i}{\delta_i}\right)^{\frac{\delta_i}{\lambda}}\right)^{\boldsymbol{\lambda}}$$

$$\left(\sum_{i=1}^{n} U_i\right)^{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^{\lambda} \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \left(\frac{U_i}{\delta_i}\right)^{\delta_i}$$

This helps us to solve geometric programming problems which is posynomial type.

Example 4.1.6: $\min f(x) = 40x_1^{-1}\, x_2^{-1} x_3^{-1} + 40x_1 x_3$

$s.t. \quad 4x_1x_2 + 2x_2x_3 \leq 8$

$x_i > 0$

Solution: we can rewrite as

$$\min f(x) = 40x_1^{-1}\, x_2^{-1} x_3^{-1} + 40x_1 x_3$$

$$s.t. \qquad \frac{1}{2}x_1x_2 + \frac{1}{4}x_2x_3 \leq 1$$

$$x_i > 0$$

$$\min f(x) = 40x_1^{-1}\, x_2^{-1}x_3^{-1} + 40x_1x_3$$

$$= U_1 + U_{2,}$$

$$\geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2}, \delta_1 + \delta_2 = 1$$

$$= \left(\frac{40x_1^{-1}\, x_2^{-1}x_3^{-1}}{\delta_1}\right)^{\delta_1} \left(\frac{40x_1x_3}{\delta_2}\right)^{\delta_2}$$

Which is taken from the objective function. But also we have from constraint function as follows:

$$\frac{1}{2}x_1x_2 + \frac{1}{4}x_2x_3 = 1$$

$$U_3 + U_3 = 1$$

$$1 = 1^{\lambda} = (U_3 + U_3)^{\lambda}$$

$$1 \geq \lambda^{\lambda} \left(\frac{U_3}{\delta_3}\right)^{\delta_3} \left(\frac{U_4}{\delta_4}\right)^{\delta_4} \text{where} \delta_3 + \delta_3 = \lambda$$

$$= \lambda^{\lambda} \left(\frac{\frac{1}{2}x_1x_2}{\delta_3}\right)^{\delta_3} \left(\frac{\frac{1}{4}x_2x_3}{\delta_4}\right)^{\delta_4}$$

Then using both inequalities from the objective function and constraind function we get that

$$\min f(x) \geq \left(\frac{40x_1^{-1}x_2^{-1}x_3^{-1}}{\delta_1}\right)^{\delta_1} \left(\frac{40x_1x_3}{\delta_2}\right)^{\delta_2} * 1$$

$$\geq \left(\frac{40x_1^{-1}x_2^{-1}x_3^{-1}}{\delta_1}\right)^{\delta_1} \left(\frac{40x_1x_3}{\delta_2}\right)^{\delta_2} \lambda^{\lambda} \left(\frac{\frac{1}{2}x_1x_2}{\delta_3}\right)^{\delta_3} \left(\frac{\frac{1}{4}x_2x_3}{\delta_4}\right)^{\delta_4}$$

$$= x_1^{-\delta_1+\delta_2+\delta_3} x_2^{-\delta_1+\delta_3+\delta_4} x_3^{-\delta_1+\delta_2+\delta_4} \left(\frac{40}{\delta_1}\right)^{\delta_1} \left(\frac{40}{\delta_2}\right)^{\delta_2} \lambda^{\lambda} \left(\frac{1}{2\delta_3}\right)^{\delta_3} \left(\frac{1}{4\delta_4}\right)^{\delta_4}$$

Then we can find the dual systems using dual variable as follows:

$$s.t.\begin{cases}-\delta_1+\delta_2+\delta_3=0\\-\delta_1+\delta_3+\delta_4=0\\-\delta_1+\delta_2+\delta_4=0\\\delta_1+\delta_2=1\end{cases}$$

Then the dual variable values are $\delta_2=\delta_3=\delta_4=\frac{1}{3},\delta_1=\frac{2}{3}$. primal variables can be solved using both equality conditions of objective and constraint functions and the answer is $x_1=\frac{1}{8},x_2=8\ and\ x_3=\frac{1}{4}$. In a standard form GP, the objective must be posynomial and it must be minimized. The equality constraints can only have the form of a monomial equal to one, and the inequality constraints can only have the form of a posynomial less than or equal to one.

Definition 1. (Pareto optimality): A point X^* in the design space Ω is said to be

i. **Weakly Pareto optimal** (or weakly efficient) if $F(X^*)$ is not strictly dominated by any other point, i.e., if there does not exist any $X\in\ \Omega\ (X\neq X^*)$, such that, $F(X)\ \neq\ F(X^*)$,and where f_i are assumed for minimization.

ii. **Pareto Optimal Solution** of a multi-objective optimization problem if there does not exist any $\in\ \Omega\ (X\neq X^*)$, such that $F(X)\ \leq\ F(X^*)$ and $F(X)\ \neq\ F(X^*)$,where f_i are assumed for minimization.

iii. **Non-dominated solution set:** Given a set of solutions, the non-dominated solution set is a set of all the solutions that are not dominated by any member of the solution set. The non-dominated set of the entire feasible decision space is called the Pareto-optimal set The boundary defined by the set of all point mapped from the Pareto optimal set is called the Pareto optimal front.

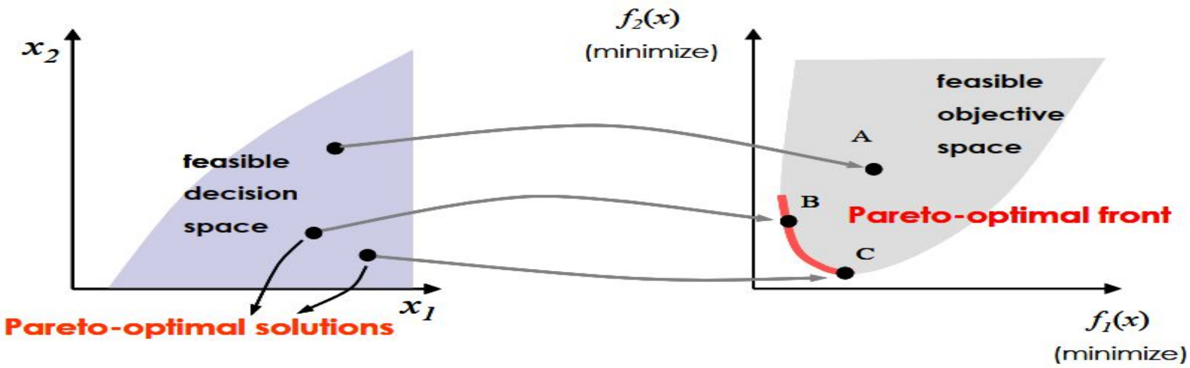


Figure 1 : Graphical Depiction of Pareto Optimal Solution

Definition 2. (Pareto Front): The collection of all efficient solutions of problem is called the efficient set. The image of the efficient set by is referred to as the Pareto front (efficient solution frontier).

Definition 3. (Weakly Pareto Optimal Solution): The point X^* is said to be weakly pareto optimal solution to the given problem (1.1) if there is no X such that $F(X) < F(X^*)$.

Definition 4. (Efficient Solution): A point $x^* \in X$ is said to be an efficient solution to MOOP with respect to the domain D if $f(x) \in k(Y, D)\ i.e\ if\ there\ is\ no\ x \in X$ such that $f(x^*) \in f(x) + D(f(x)]\backslash\{0\})$.

4.2.Fuzzy Sets

Definition 5. Let x is a fixed set. A fuzzy set A of X is an object having the form $\widetilde{\mathrm{A}} = \left\{\left(\mathrm{x}, \mu_{\mathrm{A}}(\mathrm{x})\right), \mathrm{x} \epsilon \mathrm{X}\right\}$ where the function $\mu_{\mathrm{A}}(\mathrm{x}): \mathrm{X} \to [0, 1]$ define the truth membership of the element xϵX to the set A.

Definition 6. Let f be a real valued function whose domain is a set X and f is assumed to be bounded from below by inf(f) and from above bounded by sup(f). The minimizing set is a fuzzy set M in X such that:

$$\forall x; \mu_M(x) = \frac{sup(f) - f(x)}{sup(x) - inf\ (x)}$$

From this definition we always have $\mu_{\mathrm{M}}(\mathrm{x}_0) = 1$. For all x_0 that $\mathrm{f}(\mathrm{x}_0) = \inf(\mathrm{f}) \mu_{\mathrm{M}}(\mathrm{x}_0) = 0$ for all x_0 that $\mathrm{f}(\mathrm{x}_0) = \sup(\mathrm{f})$.Clearly, the minimizing set provides essential information about the effect on the value of the objective function f of choosing values of x other than x_0. For instance for any $\bar{x} \in M$ otherthan$x_0, \mu_M(x_0) \geq \mu_M(\bar{x})\ \ if\ \ f(x_0) = inf\ (f)\ \ and\ \ \mu_M(x_0) \leq \mu_M(\bar{x}) if\ f(x_0) = sup\ (f)$.

Definition 7. Let f be a real valued function whose domain is a set X and f is assumedto be bounded from below by inf(f) and from above bounded by sup(f). The maximizing set is a fuzzy set M in X such that:

$$\forall x; \mu_M(x) = \frac{f(x) - inf\ (f)}{sup(x) - inf\ (x)}$$

From this definition we always have $\mu_M(x_0) = 1$. for all x_0 that $f(x_0) = \sup(f)$, $\mu_M(x_0) = 0$ for all x_0 that $f(x_0) = \inf(f)$.Clearly, the maximizing set provides essential information about the effect on the value of the objective function f of choosing values of x other than x_0.

4.3.Neurosophic Set

Definition 8. let X a space of points and let $x \in X$. A neutrosophic set $\widetilde{A}^n$ in X is defined by a truth-membership function $\mu_A(x)$, an indeterminacy-membership function $\sigma_A(x)$ and a false membership function $v_A(x)$ and having the form $\widetilde{A} = \{< X, \mu_A(x), \sigma_A(x), v_A(x) >/\ x \epsilon X\}$. $\mu_A(x), \sigma_A(x)$ and $v_A(x)$ are real standard or non-standard subsets of [0, 1] means

$$\mu_A(x): x \rightarrow [0, 1]$$

$$\sigma_A(x): x \rightarrow [0, 1]$$

$$v_A(x): x \rightarrow [0, 1]$$

there is no restriction on the sum $\mu_A(x), \sigma_A(x)\ and\ v_A(x)$ *so*

$$0 \leq sup\mu_A(x) + sup\sigma_A(x) + sup\ v_A(x) \leq 3$$

5. Solving MOGPOP with WS, FGP, Epsilon Constrained and NGP methods

5.1. General Formulation and Concepts of MOGPOP

The classical framework for optimization is to find the optimum value of objective functions with respect to the given constraints. All used optimization methods seek to find a single optimal solution based on a weighted sum of all objectives. If all objectives get better or worse together, then conventional approach can effectively find the optimal solution. In this case, a multi-objective optimization study should be performed which provides multiple tradeoff solutions among the objectives. A solution of multi-objective optimization problem is considered to be more a concept than a definition.

In multi-objective optimization problems, what is optimal in terms of one of the objectives is usually non-optimal for the remaining objectives. Consequently, there is no single optimal solution exist for a multi objective optimization problem. Hence we have to search for a solution which is acceptable to the decision maker. The method of optimizing systematically and simultaneously a collection of objective function is called multi-objective optimization or vector optimization. It is also defined as multi-objective optimization is the process of simultaneously optimizing multiple conflicting objectives. The difference with single-objective optimization is that MOO results in many equally efficient optimal solutions, instead of one single solution. A MOO problem consists of a set of objective functions that can be either maximized or minimized. A set of constraints limits the set of possible outcomes, known as thesolution space (Rashmi and Ojha, 2015). A general multi-objective Geomtic programming optimization problem is defined as:

Find $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)^T$ so as to

$$\left.\begin{array}{c} \min f_k(x) = \sum_{t=1}^{T_K} C_{tk} \prod_{j=1}^{n} x_j^{a_{tj}^k}, k = 1,2,\dots,p \\ \text{subject to } g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1, i = 1,2,\dots,m \\ x_j > 0, j = 1,2,\dots,n \end{array}\right\} \quad (1.1)$$

where: $\mathrm{C_{tk}} > 0$,$\mathrm{C_{it}} > 0$; $\mathrm{a_{tj}^{k}}$ and $\mathrm{a_{itj}}$ are any real numbers for all $\mathrm{i, j, k, t}$ and $\mathrm{T_k}$ = number of terms present in $\mathrm{k^{th}}$ the objective function $f_k(x)$; T_i= number of terms present in the $\mathrm{i^{th}}$ constraint.

5.2.Weighted Sum Method

Weighted sum method is used to solve a class of multi-objective geometric programming problems. Using weighted sum method we scalarize the objective functions to a single objective function at a time to find its optimal compromise solution. Weighted sum method probably is the simplest method widely used to convert a set of objectives into a single objective by multiplying each objective with weights to find the optimal compromise solution of a multi-objective optimization problem within the convex objective space.The weighting coefficients of an objective are usually chosen in proportion to the objectives relative importance in the problem. The solutions may vary significantly as the weighting coefficients changed.

In the weighting method, the idea is to associate each objective function with a weighting coefficient and minimize the weighted sum of the objectives. However, there are three difficulties with the weighted sum method. There is no satisfactory selection method to determine the weights that guarantee the final solution to be acceptable, it cannot find solutions on non-convex regions,and varying the weights may not result in an evenly distributed and accurate.There exist extensions of the weighted sum method for dealing with these three issueswhich is an improvement of weighted sum method that find solution for non-convex non linear problems is epsilon constrain method (Das and Dennis, 1997).

The weighting method is the simplest multi-objective optimization which has been widely applied to find the optimal solutions of multi-objective function within the convex objective space. If $f_1(x), f_2(x), f_3(x), f_4(x), \dots, f_K(x)$ are k objective functions for any vector $\boldsymbol{x} = (x_1, x_2, x_3, \dots, x_n)^T$ then we can define weighting method for their optimal solution as follows.

Let $\{w: w \in \mathbb{R}^n, w_j > 0, \sum_{j=1}^{n} w_j = 1\}$be set of non-negativeweights.The weighted function for the multiple objectives functions defined above in (1.1) can be defined as$min_{x=X} \sum_{j=1}^{k} w_j f_j(x)$. If the objective space of the original problem is non-convex, then the weighting method may not be capable of generating the efficient solutions on the non-convex problems. It must also be noted that the optimal solution of a weighting problem should not be used as the best optimal solution (Kao and Liu, 2004). Based on the importance of the k number of objective functions defined in (1.1) the weights $w_1, w_2, w_3, \dots, w_k$are assigned to define a new minimum type objective function $Z(w)$ which can be defined as:

$$\min Z(x) = \sum_{j=1}^{k} w_j \, f_j(x)$$

$$\text{subject to } g_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1 \, , i = 1,2,3, \dots , m$$

$$x_j > 0$$

Weighting sum method is difficult to set the weight vectors to obtain a Pareto-optimal solution in a desired region in the objective space. It cannot find certain Pareto-optimal solutions in the case of a non-convex objective space as the foowing fig.

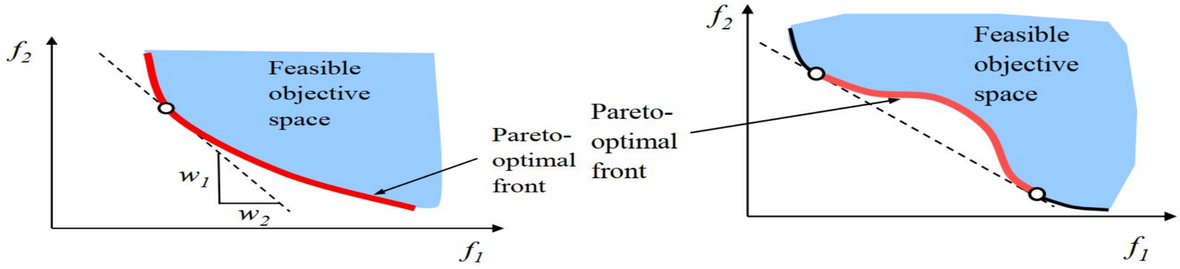


Figure 2: a) For convex case **b) For non-convex case**

Example 5.2. (Pintuet al., 2014).

$$\text{Min} f_1(x) = x_1^{-1} x_2^{-2}$$

$$\text{Min } \ f_2(x) = 2x_1^{-2} x_2^{-3}$$

$$\text{s.t. } \ x_1 + x_2 \leq 1$$

$$x_i > 0. \forall i$$

Solution:

The weighting problem corresponding to the above problem is:

$$\text{Min f(x)} = \sum_{k=1}^{p} w_k \, f_k(x) = w_1 f_1 + w_2 f_2$$

$$= w_1 [x_1^{-1} x_2^{-2}] + w_2 [2x_1^{-2} x_2^{-3}]$$

$$= w_1 x_1^{-1} x_2^{-2} + 2w_2 x_1^{-2} x_2^{-3}$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$x_i > 0 . \forall i$$

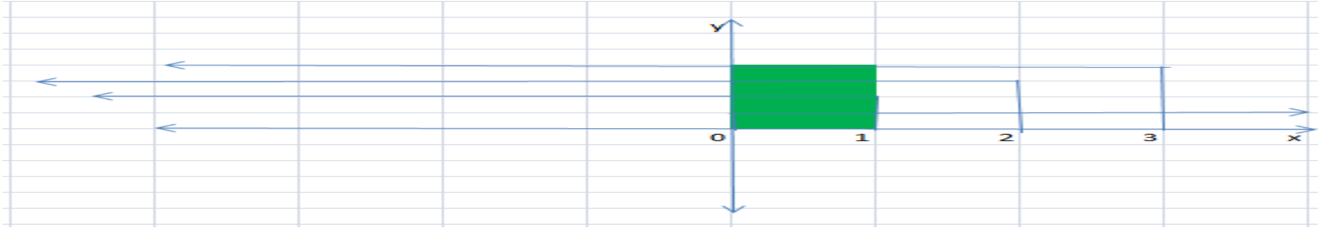
$$w_i > 0\ , i = 1{,}2\ and w_1 + w_2 = 1$$

We can identify the degree of difficulty i.e. DD=4-2-1=1, this implies that the problem has many solution means it has the different numbers of equations and decision variable. Now to solve the problem, we use AM-GM inequality relation.

$$\left.\begin{aligned}
\mathrm{Min} f_1(x) &\geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}\left(\frac{U_4}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}\\
&= \left(\frac{w_1 x_1^{-1} x_2^{-2}}{\delta_1}\right)^{\delta_1}\left(\frac{2w_2 x_1^{-2} x_2^{-3}}{\delta_2}\right)^{\delta_2}\left(\frac{x_1}{\delta_3}\right)^{\delta_3}\left(\frac{x_2}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}\\
&= \left(\frac{w_1}{\delta_1}\right)^{\delta_1}\left(\frac{2w_2}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}\\
s.t. &\begin{cases} -\delta_1 - 2\delta_2 + \delta_3 = 0 \\ -2\delta_1 - 3\delta_2 + \delta_4 = 0 \\ \delta_1 + \delta_2 = 1 \\ \delta_3 + \delta_4 = \boldsymbol{\lambda} \end{cases}
\end{aligned}\right\}$$

Let us express each dual variable in terms of one variable and let $\delta_1 = r$,$\delta_2 = 1 - r$,$\delta_3 = 2 - r$ and $\delta_4 = 3 - r$ we should know the domain of the dual variable $\delta_i > 0$

$\delta_1 = r > 0, \delta_2 = 1 - r > 0, \delta_3 = 2 - r > 0\ and\ \delta_4 = 3 - r > 0$. Then we get the intersection of these three inequalities as seen in the graph.



By using the value of $r \in (0,1)$ we can find different values of dual variable as follows.

Having the above value we can find primal optimal variable by using the equality conditions of the primal dual relationship as follows.

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2}$$

$$\frac{w_1 x_1^{-1} x_2^{-2}}{\delta_1} = \frac{2w_2 x_1^{-2} x_2^{-3}}{\delta_2}$$

$$\frac{w_1}{\delta_1 x_1 x_2^2} = \frac{2w_2}{\delta_2 x_1^2 x_2^3}$$

$$\frac{w_1}{\delta_1} = \frac{2w_2}{\delta_2 x_1 x_2}$$

$$x_1 x_2 = \frac{2w_2\delta_1}{w_1\delta_2}$$

$$x_1\left(\frac{\delta_4}{\delta_3 + \delta_4}\right) = \frac{2w_2\delta_1}{w_1\delta_2}$$

$$x_1 = \frac{2w_2\delta_1(\delta_3 + \delta_4)}{w_1\delta_2\delta_4}$$

$$\left(\frac{\delta_3}{\delta_3 + \delta_4}\right)x_2 = \frac{2w_2\delta_1}{w_1\delta_2}$$

$$x_2 = \frac{2w_2\delta_1(\delta_3 + \delta_4)}{w_1\delta_2\delta_3}$$

$$\frac{U_3}{\delta_3} = \frac{U_4}{\delta_4} = \frac{U_3 + U_4}{\delta_3+\delta_4} = \frac{1}{\delta_3+\delta_4}$$

$$\frac{U_3}{\delta_3} = \frac{1}{\delta_3+\delta_4}$$

$$\frac{x_1}{\delta_3} = \frac{1}{\delta_3+\delta_4}$$

$$\boldsymbol{x_1} = \frac{\boldsymbol{\delta_3}}{\boldsymbol{\delta_3}+\boldsymbol{\delta_4}}$$

And also we can compute the second primal variable interms of dual variables as follows

$$\frac{U_4}{\delta_4} = \frac{1}{\delta_3+\delta_4}$$

$$\frac{x_2}{\delta_4} = \frac{1}{\delta_3+\delta_4}$$

$$\boldsymbol{x_2} = \frac{\boldsymbol{\delta_4}}{\boldsymbol{\delta_3}+\boldsymbol{\delta_4}}$$

Now from the above, we get the optimal primal values of x_1 *and* x_2 using dual constants for the problem and gives by the following table.

Table 1: Optimal solutions obtained by primal and dual optimal variables in example 5.2

Optimal dual Value				Pri. Opti. Value		Obj. functiona value		
δ_1	δ_2	δ_3	δ_4	x_1	x_2	$f_1(x)$	$f_2(x)$	$f_1 + f_2$
0.1000	0.900	1.900	2.900	0.395833	0.60416	6.9212	57.8829	64.8041
0.2000	0.800	1.800	2.800	0.3913	0.60869	6.8975	57.9191	64.8166
0.3000	0.700	1.700	2.700	0.386363	0.6136	6.8743	57.9940	64.8683
0.4000	0.600	1.600	2.600	0.38095	0.6190	6.8509	58.1062	64.9571
0.5000	0.500	1.500	2.500	0.375	0.625	6.8267	58.2542	65.0809
0.6000	0.400	1.400	2.400	0.3684	0.63157	6.8051	58.4966	65.3017
0.6499	0.35000	1.350001	2.3500	0.3649	0.6351	6.7942	58.6350	65.4292
0.6492	0.35084	1.350839	2.3508	0.364926	0.64916	6.7943	58.533	65.3300
0.9416	0.05843	1.058429	2.0584	0.339582	0.66042	6.751	60.212	66.9600
0.1746	0.82541	1.825408	2.8254	0.392492	0.60751	6.903	57.904	64.8000
0.1800	0.8200	1.8200	2.8200	0.3922	0.6077	6.904	57.937	64.8082
0.2000	0.8000	1.8000	2.8000	0.3913	0.6086	6.901	57.945	64.8463
0.9000	0.1000	1.1000	2.1000	0.34375	0.65625	6.756	59.887	66.6442
0.9500	0.0500	1.0500	2.0500	0.3387	0.66129	6.751	60.287	67.0385
0.9600	0.0400	1.0400	2.0400	0.33766	0.66233	6.751	60.374	67.1247
0.9700	0.0300	1.0300	2.0300	0.3366	0.66339	6.751	60.460	67.2105
0.9900	0.0100	1.0100	2.0100	0.3344	0.66556	6.751	60.665	67.4158
0.9990	0.0010	1.0010	2.0010	0.3334	0.6665	6.752	60.774	67.5240

From above table 3 we can say that the optimal values of the given objective function is $f_1(x) = 6.903,\ f_2(x) = 57.904$ with primal variable and dual variables $x_1 = 0.392492, x_2 = 0.60751, \delta_1 = 0.1746, \delta_2 = 0.825408, \delta_3 = 1.825408, \delta_4 = 2.8254$ respectively.

5.3. Fuzzy Programming Method

Fuzzy set theory introduced by Zadeh in 1965 which is a generalization of classical set theory to understand the uncertainty and vagueness in the complexity of the problems. Fuzzy programming Problem due to Zimmermann (1990) based on the concept given by Bellman and

Zadeh (1965), has been successfully applied to solve various types of multi-objective decision making problems such as engineering design and maintenance, production planning and control, transportation, water resource management, managerial decision making and scheduling problems. A fuzzy set is associated with its membership function which is defined from its elements to the interval $[0,1]$plays an important role in solving multi-objective decision making problems. As there are several types of fuzzy membership functions, a suitable membership function is to be selected to solve the real world multi-objective mathematical programming problems. The following steps are used for solving a multi-objective optimization problem with a linear membership function by Geometric programming technique to find an optimal compromise solution. We use some procedures to solve the given (1.1) problem by fuzzy method.

Step-1: we can solve the MOGPP as a single objective GP problem using only one objective at a time and ignoring the others. These solutions are known as ideal solution. Choose one of the objective functions $f_k(x), k = 1,2,\dots,p$and solve it as a single objective Geometric programming problem subject to the constraints (1.1) and by using Geometric programming algorithms. Let $X = (X_1, X_2, \dots, X_P)$be the respective optimal solution for p differentGeometric programming problems. It is assumed that at least two of these ideal solutions are different $(f_k(x), k = 1,2,\dots,p)$ and has the different bound values. If all the optimal solutions $X^* = X_1 = X_2 = \cdots = X_P$are same then stop and $X^*$is the optimal solution. But If at least two of these ideal solutions are different optimal solutions we go to next step.

Step-2: From the results of step-1, we evaluate all these p objective functions$f_k(x), k = 1,2,\dots,p$, at all these p ideal solutions such as $X = (X_1, X_2 \dots, X_P)$ and determine the corresponding values for every objective functions at each solution derived.

Step-3: Pay-off matrix can be formulated and also used for finding lower and upper bounds of p objective function$f_k(x),\ k = 1,2,\dots,p$.

$$\begin{bmatrix} f_1(X_1) & f_2(X_1)\dots f_k(X_1) & \\ f_1(X_2) & f_2(X_2)\dots f_k(X_2) & \\ f_1(X_3) & f_2(X_3)\dots f_k(X_3) & \\ . & . \quad\quad . & \\ . & . \quad\quad . & \\ . & . \quad\quad . & . \\ f_1(X_k) & f_2(X_k)\dots f_k(X_k) & \end{bmatrix}$$

Here $X = (X_1, X_2 \dots, X_P)$ are the ideal solution of the $f_1(X), f_2(X), f_3(X), \dots, f_k(X)$ respectively. Hence we can write lower and upper bounds of each objective functions by using the above pay-off matrix. $U_k = \max\{f_k(X_1), f_k(X_2), f_k(X_3), f_k(X_4), \dots, f_k(X_k)\}, k = 1,2,3, \dots, p$and also lower bound can be written as $L_k = \min\{f_k(X_1), f_k(X_2), f_k(X_3), f_k(X_4), \dots, f_k(X_k)\}, k = 1,2,3, \dots, p$. Generally, it can be written as the best value L_k (minimum value) and the worst value U_r (maximum value) of each objective function $f_k(X)$ such that $L_k \leq f_k(X) \leq U_k, k = 1,2,3, \dots, p$. this lower and upper baunds of each objective function helps us to find the membership functions.

Step-4: Define a fuzzy membership function $\mu_k(x)$for the k^{th}objective function$f_k(X)$ as:

$$\mu_k(x) = \begin{cases} 1 & if\ f_k(x) \leq L_k \\ \dfrac{U_k - f_k(x)}{U_k - L_k} & if\, L_k \leq f_k(x) \leq U_k \\ 0 & if\ f_k(x) > U_k \end{cases}$$

Where $L_k \neq U_k, k = 1,2,3, \dots, p$ but if $L_k = U_k$, define $\mu_k(x)$=1 for any value of k. Now we can maximize the membership function $\mu_k(x), k = 1,2,3, \dots, p$ subject to the constraints in equation (1.1). We can describe the given membership function graphically.

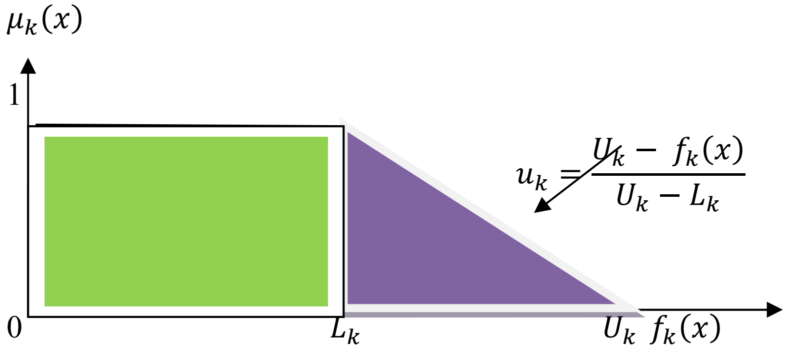


Figure 3: Membership function for minimization problem

The above fig (1) shows us the graphical description of the membership functions which is in the interval between lower and upper bound of each objective functions used for finding optimal value of the objective function. According to max-addition operator, we can describe it as:

$$\left.\begin{array}{r} \text{maximize } \sum_{k=0}^{p} \mu_k\big(f_k(x)\big) = (\mu_1(x) + \mu_2(x) + \mu_3(x) + \cdots + \mu_p(x)) \\ \text{subject to } g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1\,, i = 1,2,\ldots,m \\ \mu_k(x) = \dfrac{U_k - f_k(x)}{U_k - L_k} \end{array}\right\} \quad (1.4)$$

$0 \leq \mu_k(x) \leq 1, x_j > 0, k = 1,2,\ldots,p$. This equation (1.2) is a single objective geometric programming problem and we can solve by AM-GM inequality way of geometric programming problems.

Example 5.3. (Pintuet al.,2014).

$$\text{Min} f_1(x) = x_1^{-1} x_2^{-2}$$

$$\text{Min } f_2(x) = 2x_1^{-2} x_2^{-3}$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$x_i > 0. \forall i$$

Solution: In order to solve the above multi-objective geometric programming problems by fuzzy method, we should follow the above procedures. We divide the problem into two sub problems to find the ideal solutions which helps us to know the lower and upper bounds of the objective functions with the ideal solutions $x_i, i = 1,2,3$. This lower and upper bounds of the objective function helps to find the true membership functions.

Sub-problem one :

$$\text{Min} f_1(x) = x_1^{-1} x_2^{-2}$$
$$\text{s.t. } x_1 + x_2 \leq 1$$
$$x_i > 0. \forall i$$

In order to find the ideal solutions of this, we can use AM-GM inequality relationship. Here we can find two equations from objective function and constrained function then we combines together which is equivalent to dual problem.

$$\text{Min } f_1(x) \geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \left(\frac{U_3}{\delta_3}\right)^{\delta_3} (\delta_2+\delta_3)^{(\delta_2+\delta_3)}, \delta_1 = 1, \qquad \delta_2+\delta_3 = \boldsymbol{\lambda}$$

$$= \left(\frac{x_1^{-1}x_2^{-2}}{\delta_1}\right)^{\delta_1} \left(\frac{x_1}{\delta_2}\right)^{\delta_2} \left(\frac{x_2}{\delta_3}\right)^{\delta_3} (\boldsymbol{\lambda})^{(\lambda)}$$

$$= \left(\frac{x_1^{-1}x_2^{-2}}{\delta_1}\right)^{\delta_1} \left(\frac{x_1}{\delta_2}\right)^{\delta_2} \left(\frac{x_2}{\delta_3}\right)^{\delta_3} (\boldsymbol{\lambda})^{(\lambda)}$$

$$= \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{1}{\delta_3}\right)^{\delta_3} (\boldsymbol{\lambda})^{(\lambda)}$$

$$s.t. \begin{cases} -\delta_1+\delta_2 = 0 \\ -2\delta_1+\delta_3 = 0 \\ \delta_1 = 1 \\ \delta_2+\delta_3 = \boldsymbol{\lambda} \end{cases}$$

From the above dual constrained we can solve the optimal dual variable which is $\delta_1 = 1, \delta_2 = 1, \delta_3 = 2$ and optimal primal Variable can be obtained by using the equality conditions of primal-dual relation using AM-GM inequality relationship as follows. The first equality condition is taken from the objective function as follows:

$$\frac{U_1}{\delta_1} = \frac{f_1}{1} = f_1 = U_1 = x_1^{-1}x_2^{-2}$$

$$\frac{x_1^{-1}x_2^{-2}}{1} = f_1$$

The second condition is taken from the constrained function as follows:

$$\frac{U_2}{\delta_2} = \frac{U_3}{\delta_3} = \frac{U_2+U_3}{\delta_2+\delta_3} = \frac{1}{1+2} = \frac{1}{3}$$

$$\frac{U_2}{\delta_2} = \frac{1}{3} \; and \frac{U_3}{\delta_3} = \frac{1}{3}$$

$$\frac{x_1}{1} = \frac{1}{3}, x_1 = \frac{1}{3}$$

$$\frac{x_2}{\delta_3} = \frac{x_2}{2} = \frac{1}{3}, then \; x_2 = \frac{2}{3}, X_1 = (x_1, x_2) = (\frac{1}{3}, \frac{2}{3})$$

Therefore we can find the optimal values of the objective function using both primal and dual variables.

$$f_1(x) = x_1^{-1}x_2^{-2}$$
$$= (x_1)^{-1}(x_2)^{-2}$$
$$= \left(\frac{1}{3}\right)^{-1}(\frac{2}{3})^{-2}$$
$$= \frac{27}{4}$$

Thus, $f_1(x) = 6.75$ and let us check the functional values of objective function one by using optimal dual value as follows.

$$\text{Min } f_1(x) \geq \left(\frac{1}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}(\lambda)^{(\lambda)}$$
$$= \left(\frac{1}{1}\right)^{1}\left(\frac{1}{1}\right)^{1}\left(\frac{1}{2}\right)^{2}(3)^{(3)}$$
$$= 1 * 1 * \frac{1}{2}(3)^{(3)}$$
$$= \frac{27}{4} = 6.75$$

Now fron the AM-GM inequality relationship we can understand that the maximum of the dual objective function is the minimum of the primal objective function which is equal to 6.75.

Sub –Problem Two:

$$\text{Min } f_2(x) = 2x_1^{-2}x_2^{-3}$$
$$\text{s.t. } x_1 + x_2 \leq 1$$
$$x_i > 0. \forall i$$

We should follow the same procedure as the sub problem one above using the primal-Dual relation using AM-GM inequality.

$$\text{Min } f_1(x) \geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}(\delta_2+\delta_3)^{(\delta_2+\delta_3)}, \delta_1 = 1, \qquad \delta_2+\delta_3 = \boldsymbol{\lambda}$$
$$= \left(\frac{2x_1^{-2}x_2^{-3}}{\delta_1}\right)^{\delta_1}\left(\frac{x_1}{\delta_2}\right)^{\delta_2}\left(\frac{x_2}{\delta_3}\right)^{\delta_3}(\boldsymbol{\lambda})^{(\lambda)}$$
$$= \left(\frac{2}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}(\boldsymbol{\lambda})^{(\lambda)}$$

$$s.t. \quad \begin{cases} -2\delta_1 + \delta_2 = 0 \\ -3\delta_1 + \delta_3 = 0 \\ \delta_1 = 1 \\ \delta_2 + \delta_3 = \boldsymbol{\lambda} \end{cases}$$

From above dual constrained systems of equation we can say that the problem has unique solution because the degree of difficulty is zero (DD = 3-2-1=0). Now the values of the dual variable are $\delta_1 = 1, \delta_2 = 2$ and $\delta_3 = 3$ then we can find the primal variable by using the equality conditions of both constraint and objective functions.

$$\frac{U_1}{\delta_1} = \frac{f_1}{1} = f_1 = U_1 = 2x_1^{-2}x_2^{-3}$$

$$2x_1^{-2}x_2^{-3} = f_1$$

The second condition is taken from the constrained function as follows:

$$\frac{U_2}{\delta_2} = \frac{U_3}{\delta_3} = \frac{U_2 + U_3}{\delta_2 + \delta_3} = \frac{1}{2+3} = \frac{1}{5}$$

$$\frac{U_2}{\delta_2} = \frac{1}{5} \, and \, \frac{U_3}{\delta_3} = \frac{1}{5}$$

$$\frac{x_1}{2} = \frac{1}{5}, this\ implies\ that\ x_1 = \frac{2}{5}$$

$$\frac{x_2}{\delta_3} = \frac{x_2}{3} = \frac{1}{5}, \text{this implies that } x_2 = \frac{3}{5}$$

Therefore, $X_2 = (x_1, x_2) = (\frac{2}{5}, \frac{3}{5})$. Then the optimal values of the objective function using the primal variable value is$f_2(x) = 2x_1^{-2}x_2^{-3} = 2\left(\frac{2}{5}\right)^{-2}(\frac{3}{5})^{-3} = 57.8703, \; f_2(x) = 57.8703.$

And Let us check the second objective function optimal value by using the dual variable obtained above as follows.

$$\text{Min } f_1(x) \geq \left(\frac{2}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}(\lambda)^{(\lambda)},$$

$$= \left(\frac{2}{1}\right)^{1}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{3}\right)^{3}(5)^{(5)},$$

$$= 2 * \frac{1}{4} * \frac{1}{27} * 3125,$$

$$= \frac{3125}{4 * 27} = 57.8703$$

Therefore, we can find each objective functional values at each primal decision variables and the use for determining the pay-off matrix which helps to find upper and lower bounds of each objective function.

$$f_1(X_1) = 6.75$$
$$f_1(X_2) = 6.94$$
$$f_2(X_2) = 57.8703$$
$$f_2(X_1) = 60.75$$

Now we can construct the pay-off matrix by using the ideal solutions of the two objective functions which used to know the lower and upper bound of each objective functions.

$\begin{bmatrix} f_1(X_1) & f_2(X_1) \\ f_1(X_2) & f_2(X_2) \end{bmatrix} = \begin{bmatrix} 6.75 & 60.75 \\ 6.94 & 57.87 \end{bmatrix}$, From the pay-off matrix the lower and upper bound can be written as follows.

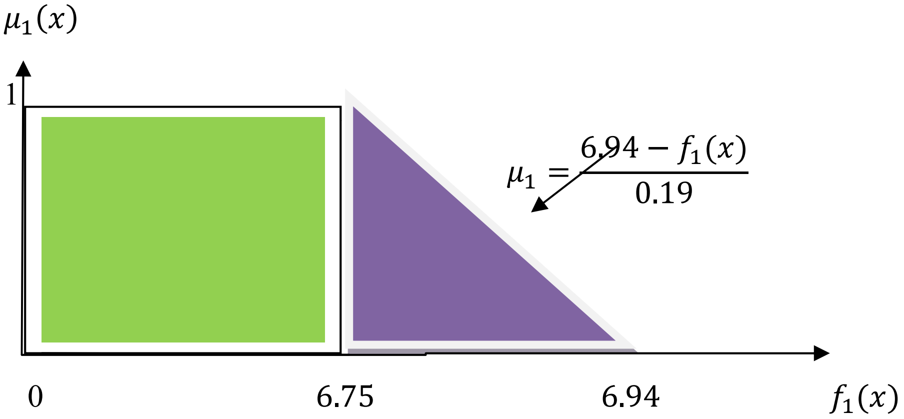
$$L_1 = 6.75 \leq f_1(X) \leq 6.94 = U_1$$
$$L_2 = 57.87 \leq f_2(X) \leq 60.75 = U_2$$

The above computed lower and upper bounds helped us to determine the membership function and we can be written as follows:

$$\mu_1(x) = \begin{cases} 1 & if\, f_1(x) \leq 6.75 \\ \dfrac{6.94 - f_1(x)}{0.19}, & if\ 6.75 \leq f_1(x) \leq 6.94 \\ 0, if\ f_1(x)) > 6.94 & \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & if\, f_2(x) \leq 57.87 \\ \dfrac{60.75 - f_2(x)}{2.88}, & if\ 57.87 \leq f_2(x) \leq 60.75 \\ 0, if\ f_2(x)) > 60.75 & \end{cases}$$

We can sketch the graph of this two membership function $\mu_1(x), \mu_2(x)$ as follows.



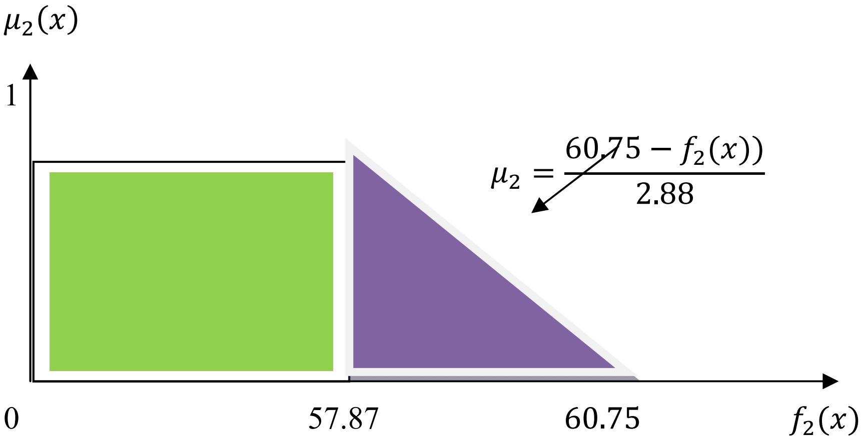


Figure 4: Values of membership functions for minimization problem

Now we use the Max-addition operator to transform the membership functions to single objective geometric programming problem with the original constrained function.

$$max \sum_{k=1}^{2} \mu_k\big(f_k(x)\big)$$

$$max\ \mu_1\big(f_1(x)\big) + \mu_2\big(f_2(x)\big) = \frac{6.94 - f_1(x)}{0.19} + \frac{60.75 - f_2(x))}{2.88})$$

$$= \frac{6.94}{0.19} - \frac{f_1(x)}{0.19} + \frac{60.75}{2.88} - \frac{f_2(x)}{2.88}$$

$$= \frac{6.94}{0.19} + \frac{60.75}{2.88} - (\frac{f_1(x)}{0.19} + \frac{f_2(x)}{2.88})$$

For maximizing the above max-addition operator problem, we minimize membership functions. only by cancelling the constant term with the original constrained.

$$\text{Min } S(x) = \frac{f_1(x)}{0.19} + \frac{f_2(x)}{2.88}$$

$$= \frac{x_1^{-1}x_2^{-2}}{0.19} + \frac{2x_1^{-2}x_2^{-3}}{2.88}$$

$$= 5.269x_1^{-1}x_2^{-2} + 0.699x_1^{-2}x_2^{-3}$$

Then we can write general problem as:

$$\text{Min } S(x) = 5.269x_1^{-1}x_2^{-2} + 0.699x_1^{-2}x_2^{-3}$$

$$\text{s.t. } x_1 + x_2 \ \leq 1$$

$$x_i > 0. \forall i$$

The above problem is SOGPP then we can solve by primal-dual relationship or AM-GM inequality relation as follows.

$$\operatorname{Min} f_1(x) \geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}\left(\frac{U_4}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}$$

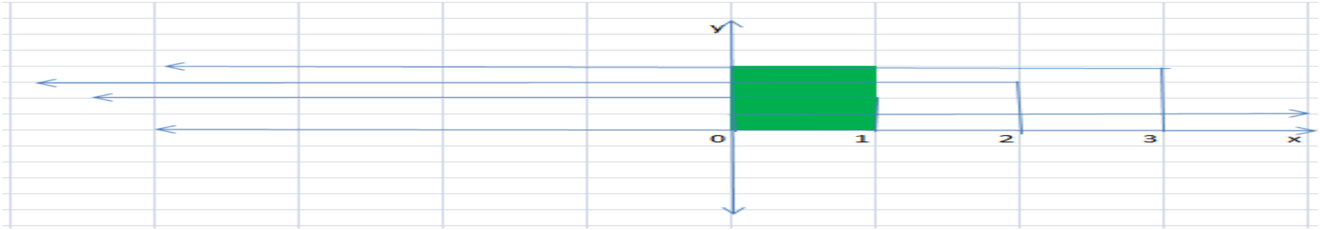
$$=\left(\frac{5.269x_1^{-1}x_2^{-2}}{\delta_1}\right)^{\delta_1}\left(\frac{0.699x_1^{-2}x_2^{-3}}{\delta_2}\right)^{\delta_2}\left(\frac{x_1}{\delta_3}\right)^{\delta_3}\left(\frac{x_2}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}$$

$$=\left(\frac{5.269}{\delta_1}\right)^{\delta_1}\left(\frac{0.699}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}$$

$$s.t.\quad \begin{cases} -\delta_1-2\delta_2+\delta_3=0\\ -2\delta_1-3\delta_2+\delta_4=0\\ \delta_1+\delta_2=1\\ \delta_3+\delta_4=\boldsymbol{\lambda}\end{cases}$$

Let us express each dual variable in terms of one variable and let $\delta_1=r$,$\delta_2=1-r$,$\delta_3=2-r$ and $\delta_4=3-r$ we should know the domain of the dual variable $\delta_i>0$

$\delta_1=r>0,\delta_2=1-r>0,\delta_3=2-r>0\ and\ \delta_4=3-r>0$. Then we get the intersection of these three inequalities as seen in the graph.



By using the value of $r\in(0,1)$ we can find different values of dual variable as follows.

Having the above value we can find primal optimal variable by using the equality conditions of the primal dual relationship as follows.

$$\delta_1>0,\delta_2=1-\delta_1>0,\delta_3=2-\delta_1>0\ and\ \delta_4=3-\delta_1>0$$

$$\operatorname{Min} Z(x)=\left(\frac{5.269}{\delta_1}\right)^{\delta_1}\left(\frac{0.699}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)}$$

$$=\left(\frac{5.269}{\delta_1}\right)^{\delta_1}\left(\frac{0.699}{1-\delta_1}\right)^{1-\delta_1}\left(\frac{1}{2-\delta_1}\right)^{2-\delta_1}\left(\frac{1}{3-\delta_1}\right)^{3-\delta_1}(5-2\delta_1)^{(5-2\delta_1)}$$

$$0<\delta_1<1$$

The optimal value canbeobtained by differention as follows:

$$\frac{d(Z(\delta_1))}{d(\delta_1)}=0$$

$$5.263(1-\delta_1)(2-\delta_1)(3-\delta_1) = 0.687\delta_1(5-2\delta_1)^2$$

Therefore from above fuzzy geometric programming method we get that both primal and dual variables and the corresponding optimal functional values. The dual variables are $\delta_1 = 0.7035507, \delta_1 = 0.2964493, \delta_1 = 1.296449\ and \delta_1 = 2.296449$ and the then we can also find the primal values using the dual variables with the following result. $x_1 = 0.360836\ and\ x_2 = 0.6391634$ the corresponding objective functional value is $f_1(x_1, x_2) = 58.82652\ and\ f_2(x_1, x_2) = 6.783684.$

5.4.Epsilon Constrained Method

A method which overcomes some of the convexity problems of the weighted sum technique is known as epsilon constraint method. This method involves minimizing a primary objective and expressing the other objectives in the form of inequality constraints.

The constraint method was proposed by Haimes *et al.* (1971) for generating optimal solutions for the multi-objective optimization problem. We have applied constraint method to solve a class of multi-objective Geometric programming problems. Using epsilon constraint method, we can optimize one of the objective function at a time where other objectives are kept in the constraint. This method is found more suitable than other generating methods used for obtaining optimal solutions. After obtaining lower and upper bounds of each objective function with the given constraints, we have generated a set of optimal solution.We use ε-constraint to solve multi-objective Geometric programming problems for searching a compromise solution.

One difficulty about this method is how to choose ε_m values; you do not know before hand what the best ε values will be. Thus it requires more user input to find solutions in non-convex regions, but the Constraint method has also several advantages over the Weighted Sum Method: The Weighted Sum Method only finds extreme points for linear models, thus a lot of runs are redundant because they result in the same solution. The Constraint Method is able to produce non-extreme efficient solutions, the Weighted Sum Method cannot find solutions for non-convex regions, while the epsilon Constraint Method does not suffer from this pitfall and it could Constraint Method can control the number of efficient solutions, while this is not so easy for the Weighted Sum Method.

There is no specific mathematical proof for the convergence of the optimal solutions of the multi-objective mathematical programming problem. However the decision maker try to find out the most compromise solutions by using some of the existing methods like fuzzy programming, NGP and weighting methods. In the present work we have used epsilon constraint method to find the optimal solution. Here, we have adopted the following steps to show the set of optimal solutions are converging to certain point.

Step 1: First find the bounds of the objective functions $f_k(x), k = 1,2,3, \dots, p$ with the help of obtained ideal solutions by using Geometric programming algorithms. Let $\mathrm{L_k}$ and $\mathrm{U_k}$ are the best and worst values of $f_k(x)$ *i.e.* $L_k \le f_k(x) \le U_k$, $k = 1,2,3, \dots, p$.
Step 2: Let ε_k , be a point in the interval such that $L_k \le \varepsilon_k \le U_k$, $k = 1,2,3, \dots, p$.
Step 3: Changing the value of ε_k in the interval $[\mathrm{L_k}, \mathrm{U_k}]$, it generate a set of optimal solution.
Step 4: Compare the optimal solution with the solution obtained by fuzzy programming method.
Step 5. If the optimal solution obtained in Step 3 is equal to the optimal compromise solution obtained in Step 4, then stop and accept the pareto optimal solution of the problem. This indicate that the set of solution generated by ε- constraint method converges to this particular solution. However, the decision maker has the choice to choose his/her solution from the set of solution according to their satisfaction.The solution of dual variable not only gives the primal optimal solution but also provides a relationship between maximum and minimum value of the objective function.

The epsilon constraint method used for generating optimal solutions for the multi-objective optimization problem. Thismethod generates solutions of multi-objective optimization problemsby considering one objective function at a time as primary one and convertingthe remaining objective functions as constraints. In other words, it minimizes one objective function and simultaneously maintains the maximum acceptability for other objective function. The *epsilon* constraint method is defined as:

$$\left.\begin{array}{c} \min f_k(x), k = 1,2,3,\dots,p \\ f_j(x) \le \varepsilon_j, j = 1,2,\dots,p \\ \text{subject to } g_i(x) \le 1 \\ L_j \le \varepsilon_j \le U_j \\ L_j = \min f_j(X), j = 1,2,3,\dots,p, \\ U_j = \max f_j(X), j = 1,2,3,\dots,p, \\ j \ne k \\ i = 1,2,\dots,m \\ x_j > 0 \end{array}\right\} \qquad 1.5$$

The ε vector has to be chosen carefully so that it is within the minimum or maximum values of the individual objective function.

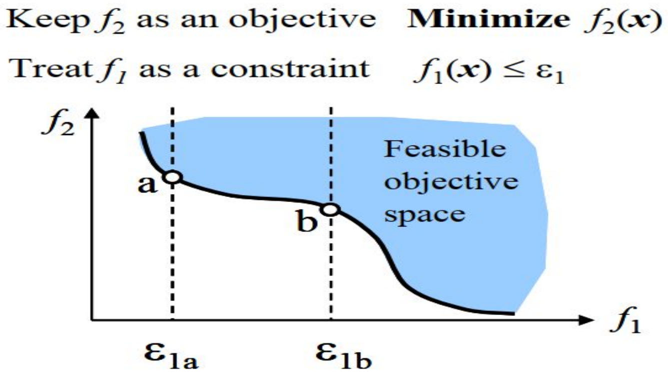


Figure 5: fixing epsilon value between lower and upper baunds of function

Example 5.4. (Pintuet al., 2014)

Solve the following multi-objective geometric programming problems by epsilon constrained method.

$$\text{Min} f_1(x) = x_1^{-1} x_2^{-2}$$

$$\text{Min } f_2(x) = 2x_1^{-2} x_2^{-3}$$

$$\text{s.t. } x_1 + x_2 \le 1$$

$$x_i > 0. \forall i$$

Solution:

In order to solve theabove problem by the method mentioned, we follow the procedures described in above. First we divide the problem into two and then find the lower and upper bounds of each objective function which helps to know the range of ε_k , $k = 1,2,3,\ldots,p$ like

$\varepsilon_1 \in [L_1,\ U_1\ ]$, $\varepsilon_2 \in [L_2,\ U_2\ ]$,$\varepsilon_3 \in [L_3,\ U_3\ ]$,…,$\varepsilon_k \in [L_k,\ U_k\ ]$.

Sub-Problem One: Here we should find first the ideal solutions for each objective function with common constraint function which helps us to find the bounderies of the epsilon.

$$\text{Min} f_1(x) = x_1^{-1}x_2^{-2}$$
$$\text{s.t.}\ \ x_1 + x_2\ \leq 1$$
$$x_i > 0. \forall i$$

We can solve by using primal dual relationship and the solution is already computed in fuzzy method as $X_1 = (x_1, x_{2)=}(\frac{1}{3},\frac{2}{3})$ and the we compute the optimal values of the objective functions as $f_1(x) = x_1^{-1}x_2^{-2} = (x_1)^{-1}(x_2)^{-2} = \left(\frac{1}{3}\right)^{-1}(\frac{2}{3})^{-2} = \frac{27}{4}$, this implies that $f_1(x) = 6.75$.

$$f_1(X_1) = 6.75$$
$$f_2(X_1) = 60.75$$

Sub-Problem Two:

$$\text{Min}\ \ f_2(x) = 2x_1^{-2}x_2^{-3}$$
$$\text{s.t.}\ \ x_1 + x_2\ \leq 1$$
$$x_i > 0. \forall i$$

Solution: This sub-problem two can be solved with the same procedure as sub-problem one and the solution is already computed in the fuzzy method above as $X_2 = (x_1, x_2) = (\frac{2}{5},\frac{3}{5})$ and the functional values of the objective functions are $f_2(x) = 57.8703$. Now by using the above ideal solutions we can identify the lower and upper bounds of the objective functions $f_1(X_2)$ and $f_2(X_2)$ as seen below.We can also find the optimal values of each objective function on each optimal primal variable.

$$f_1(X_1) = 6.750$$
$$f_1(X_2) = 6.940$$
$$f_2(X_2) = 57.870$$
$$f_2(X_1) = 60.750$$

$L_1 = 6.75 \ \leq f_1(X) \leq 6.94 = U_1, L_2 = 57.8703 \leq f_2(X) \leq 60.75 \quad = U_2$. We can rewrite it as $L_1 = 6.75 \ \leq \varepsilon_1 \leq 6.94 = U_1,\ L_2 = 57.8703 \leq \varepsilon_2 \leq 60.75 \quad = U_2$. Now we can formulate the above multi-objective geometric programming problem as two different problems using epsilon constrained method.

1. Primal problem (i) using $\boldsymbol{\varepsilon}$ −constained method: When we formulate this problem we using epsilon constraint method, we can optimize one of the objective functions at a time where other objectives are kept in the constraint.

$$\begin{gathered} \text{Min} f_1(x) = x_1^{-1}x_2^{-2} \\ \text{s.t.}\ \ x_1 + x_2 \ \leq 1 \\ 2x_1^{-2}x_2^{-3} \leq \varepsilon_2 \\ x_i > 0. \forall i \end{gathered}$$

Solution Primal problem (i) using $\boldsymbol{\varepsilon}$ −constained method: The above problem can be rewrite as follows:

$$\begin{gathered} \text{Min} f_1(x) = x_1^{-1}x_2^{-2} \\ \text{s.t.}\ \ x_1 + x_2 \ \leq 1 \\ \frac{2x_1^{-2}x_2^{-3}}{\varepsilon_2} \leq 1 \\ x_i > 0. \forall i \end{gathered}$$

Now use primal-dual relationship or AM-GM inequality relation to solve the above problem. Here consider both objective function and constrained function to have the inequality.

$$\text{Min} f_1(x) \geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \left(\frac{U_3}{\delta_3}\right)^{\delta_3} \left(\frac{U_4}{\boldsymbol{\delta_4}}\right)^{\delta_4} (\boldsymbol{\lambda_1})^{\lambda_1}(\boldsymbol{\lambda_2})^{\lambda_2}$$

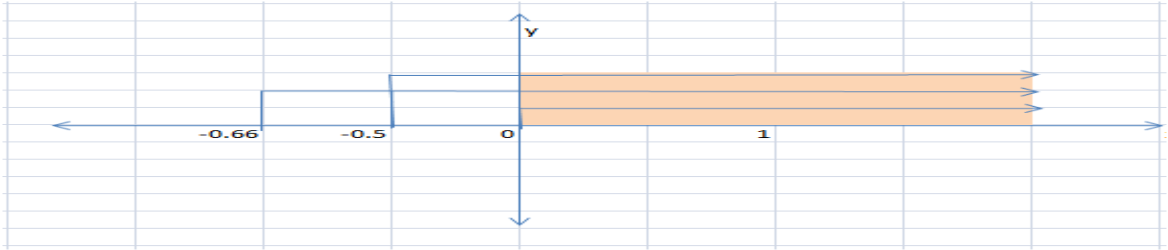
$$= \left(\frac{x_1^{-1}x_2^{-2}}{\delta_1}\right)^{\delta_1} \left(\frac{x_1}{\delta_2}\right)^{\delta_2} \left(\frac{x_2}{\delta_3}\right)^{\delta_3} \left(\frac{\frac{2x_1^{-2}x_2^{-3}}{\boldsymbol{\varepsilon_2}}}{\boldsymbol{\delta_4}}\right)^{\delta_4} (\boldsymbol{\lambda_1})^{\lambda_1}(\boldsymbol{\lambda_2})^{\lambda_2}$$

$$= \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{1}{\delta_2}\right)^{\delta_2} \left(\frac{1}{\delta_3}\right)^{\delta_3} \left(\frac{2}{\boldsymbol{\varepsilon_2}\boldsymbol{\delta_4}}\right)^{\delta_4} (\boldsymbol{\lambda_1})^{\lambda_1}(\boldsymbol{\lambda_2})^{\lambda_2}$$

$$\text{s.t.}\begin{cases} -\delta_1 + \delta_2 - 2\delta_4 = 0 \\ -2\delta_1 + \delta_3 - 3\delta_4 = 0 \\ \delta_1 = 1 \\ \delta_4 = \boldsymbol{\lambda_2} \\ \delta_2 + \delta_3 = \boldsymbol{\lambda_1} \end{cases}$$

From above systems of equation we can understand that the system has many solution which means degree of difficulty is greater than or equals to one (i.e. DD=4-2-1=1). Then we can solve the system of equation given above as follows:

Let $\delta_4 = r$ then we can express all dual variables in terms ofδ_4, then we get that $\delta_1 = 1, \delta_2 = 1 + 2r, \delta_3 = 2 + 3r \; and \; \delta_4 = r$. Then we should compute the domain of the dual variable$1 + 2r > 0,\; r > 0, 2 + 3r > 0$. Therefore we have $r > 0, r > \frac{-1}{2} \; and \; r > \frac{-2}{3}$ then we can use different values of r to find the compromise solution.



Therefore we have the domain for the dual variable above as $r \in (0, \infty)$. Then by using any values of r we can find the values of dual variable as follows.

When $r = 1, \delta_1 = 1, \delta_2 = 1 + 2r = 1 + 2 * 1 = 3, \delta_3 = 2 + 3r = 2 + 3 * 1 = 5, \delta_4 = 1$

$r = 2, \delta_1 = 1, \delta_2 = 1 + 2r = 1 + 2 * 2 = 5, \delta_3 = 2 + 3r = 2 + 3 * 2 = 8, \delta_4 = 2$

$r = 3, \delta_1 = 1, \delta_2 = 1 + 2r = 1 + 2 * 3 = 7, \delta_3 = 2 + 3r = 2 + 3 * 3 = 11, \delta_4 = 3$

$r = 4, \delta_1 = 1, \delta_2 = 1 + 2r = 1 + 2 * 4 = 9, \delta_3 = 2 + 3r = 2 + 3 * 4 = 14, \delta_4 = 4$

We can also find the primal variables by using the AM-GM inequality as follows.

$\frac{U_1}{\delta_1} = \frac{f_1}{\delta_1} = \frac{f_1}{1} = f_1$. Then we get that $U_1 = f_1 = x_1^{-1} x_2^{-2}$. From the constraint part we have equality conditions as $\frac{U_2}{\delta_2} = \frac{U_3}{\delta_3} = \frac{U_2 + U_3}{\delta_2 + \delta_3} = \frac{1}{\delta_2 + \delta_3}$, implies that $\frac{U_2}{\delta_2} = \frac{1}{\delta_2 + \delta_3}$ and then we get$\boldsymbol{x_1} = \boldsymbol{\frac{\delta_2}{\delta_2 + \delta_3}}$. But also we can find the optimal primal variable two in the same way that:

$$U_2 = \frac{\delta_2}{\delta_2 + \delta_3}$$

$$\boldsymbol{x_2} = \boldsymbol{\frac{\delta_3}{\delta_2 + \delta_3}}$$

From the constraint three we get that:

$$\frac{U_4}{\delta_4} = \frac{1}{\delta_4}$$

$$U_4 = 1$$

$$\frac{2x_1^{-2}x_2^{-3}}{\boldsymbol{\varepsilon_2}} = 1$$

$$\frac{1}{x_1^2x_2^3} = \frac{\boldsymbol{\varepsilon_2}}{2}$$

$$x_1^2x_2^3 = \frac{2}{\boldsymbol{\varepsilon_2}}$$

Now by substitution we got the following result:

$$x_1^2\left(\frac{\delta_3}{\delta_2+\delta_3}\right)^3 = \frac{2}{\boldsymbol{\varepsilon_2}}$$

$$x_1^2 = \frac{2}{\boldsymbol{\varepsilon_2}}\left(\frac{\delta_2+\delta_3}{\delta_3}\right)^3$$

$$x_1 = \left(\frac{2}{\boldsymbol{\varepsilon_2}}\left(\frac{\delta_2+\delta_3}{\delta_3}\right)^3\right)^{1/2}$$

$$\boldsymbol{x_1} = \left(\frac{\mathbf{2}}{\boldsymbol{\varepsilon_2}}\right)^{\mathbf{1/2}}\left(\frac{\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}{\boldsymbol{\delta_3}}\right)^{\mathbf{3/2}}$$

But also for the other equality condition we have as follows and used to deteriminexx_2

$$\left(\frac{\delta_2}{\delta_2+\delta_3}\right)^2 x_2^3 = \frac{2}{\boldsymbol{\varepsilon_2}}$$

$$\left(\frac{\delta_2}{\delta_2+\delta_3}\right)^2 x_2^3 = \frac{2}{\boldsymbol{\varepsilon_2}}$$

$$x_2 = \left(\frac{2}{\boldsymbol{\varepsilon_2}}\left(\frac{\delta_2+\delta_3}{\delta_2}\right)^2\right)^{1/3}$$

$$\boldsymbol{x_2} = \left(\frac{\mathbf{2}}{\boldsymbol{\varepsilon_2}}\right)^{\mathbf{1/3}}\left(\frac{\boldsymbol{\delta_2}+\boldsymbol{\delta_3}}{\boldsymbol{\delta_2}}\right)^{\mathbf{2/3}}$$

We substitute above equation to express each primal variable in terms of epsilon two and dual variable.Having the above value we can find primal optimal variable by using the equality conditions of the primal dual relationship as follows.

Table 2 : Optimal solution obtained by ε-constrained method for objective function one

Dual variable				Primal variable			Obj. Fun
δ_1	δ_2	δ_3	δ_4	x_1	x_2	ε_2	$f_1(x)$

1.000	1.00002	2.00003	0.00001	0.3333	0.666693	60.75	6.7500
1.000	1.2000	2.3000	0.10000	0.3427	0.6570	60.00	6.7613
1.000	5.0000	8.0000	2.0000	0.3847	0.6154	58.00	6.8637
1.000	201.00	302.000	100.00	0.39959	0.6004	57.871	6.9423
1.000	2001	3002	1000	0.39995	0.60007	57.871	6.9437
1.000	20001	30002	10000	0.399989	0.6000325	57.871	6.9438
1.000	7.000	11.000	3.000	0.38700	0.6092	58.50	6.9625

From this Tabe 2 we conclude that the optimal values of the objective function ranges between $\varepsilon_1 \in [57.875, 60.75]$ and compromise solution obtained by primal and dual variable are $\delta_1 = 1.000,\ \delta_2 = 1.00002, \delta_3 = 2.00003, \delta_4 = 0.00001, x_1 = 0.3333\ and\ x_1 = 0.66669\ with\ f_1(x) = 6.7500.$

Primal problem (ii) using $\boldsymbol{\varepsilon}$ −constained method: When we formulate this problem by using epsilon constraint method, we can optimize one of the objective functions at a time where other objectives are kept in the constraint.

$$\begin{gathered}
\mathrm{Min}\ \ \mathrm{f_2(x) = 2x_1^{-2}x_2^{-3}} \\
\mathrm{s.t.}\ \ \mathrm{x_1 + x_2\ \le 1} \\
\mathrm{x_1^{-1}x_2^{-2} \le \varepsilon_2} \\
\mathrm{x_i > 0.} \forall \mathrm{i}
\end{gathered}$$

Solution Primal problem (i) using $\boldsymbol{\varepsilon}$ −constained method: The above problem can be rewrite.

$$\begin{gathered}
\mathrm{Min} f_2(x) = 2x_1^{-2}x_2^{-3} \\
\mathrm{s.t.}\ \ x_1 + x_2\ \le 1 \\
\frac{x_1^{-1}x_2^{-2}}{\varepsilon_2} \le 1 \\
x_i > 0. \forall i
\end{gathered}$$

We can solve the problem using primal dual relation using AM-GM inequality as follows.

$$\mathrm{Min} f_1(x) \ge \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \left(\frac{U_3}{\delta_3}\right)^{\delta_3} \left(\frac{U_4}{\boldsymbol{\delta_4}}\right)^{\boldsymbol{\delta_4}} (\boldsymbol{\lambda_1})^{\boldsymbol{\lambda_1}} (\boldsymbol{\lambda_2})^{\boldsymbol{\lambda_2}}$$

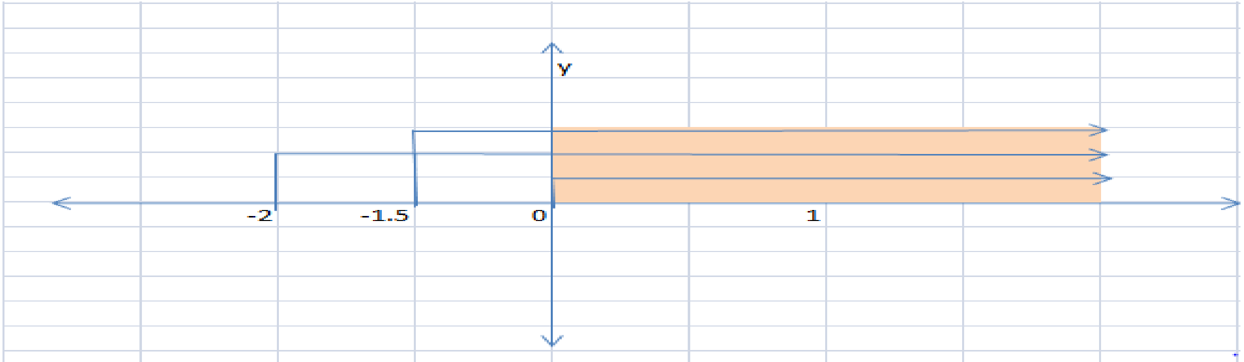
$$= \left(\frac{2x_1^{-2}x_2^{-3}}{\delta_1}\right)^{\delta_1}\left(\frac{x_1}{\delta_2}\right)^{\delta_2}\left(\frac{x_2}{\delta_3}\right)^{\delta_3}\left(\frac{\frac{x_1^{-1}x_2^{-2}}{\varepsilon_2}}{\boldsymbol{\delta_4}}\right)^{\boldsymbol{\delta_4}}(\boldsymbol{\lambda_1})^{\boldsymbol{\lambda_1}}(\boldsymbol{\lambda_2})^{\boldsymbol{\lambda_2}}$$

$$= \left(\frac{2}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{4\delta_3}\right)^{\delta_3}\left(\frac{1}{\boldsymbol{\varepsilon_2\delta_4}}\right)^{\boldsymbol{\delta_4}}(\boldsymbol{\lambda_1})^{\boldsymbol{\lambda_1}}(\boldsymbol{\lambda_2})^{\boldsymbol{\lambda_2}}$$

$$\text{s.t.}\begin{cases} -2\delta_1 + \delta_2 - \delta_4 = 0 \\ -3\delta_1 + \delta_3 - 2\delta_4 = 0 \\ \delta_1 = 1 \\ \delta_2 + \delta_3 = \boldsymbol{\lambda_1} \\ \delta_4 = \boldsymbol{\lambda_2} \end{cases}$$

From above systems of equation we can understand that the system has many solution means DD=4-2-1=1. Then we can solve the system as follows: let $\delta_4 = r$ then we can express all dual variables in terms of δ_4, then we get that:

$\delta_2 = 2 + \delta_4$and$\delta_2 = 3 + 2\delta_4$. Then we have $\delta_1 = 1,\ \delta_2 = 2 + r,\ \delta_3 = 3 + 2r\ and\ \delta_4 = r$. Then we restrict the domain of the above dual variable as: $2 + r > 0, r > 0{,}3 + 2r > 0$ Then the intersection of the above three inequalities are $r > \frac{-3}{2}, r > -2\ and\ r > 0$is $r > 0$.



Having the above values of$r \in (0, \infty)$, we get different values of dual variable.

i.e. When $r = 1, \delta_1 = 1, \delta_2 = 2 + r = 2 + 1 = 3, \delta_3 = 3 + 2r = 3 + 2 * 1 = 5, \delta_4 = 1$

$r = 2, \delta_1 = 1, \delta_2 = 2 + r = 2 + 2 = 4, \delta_3 = 3 + 2r = 3 + 2 * 2 = 7, \delta_4 = 1$

$r = 3, \delta_1 = 1, \delta_2 = 2 + r = 2 + 3 = 5, \delta_3 = 3 + 2r = 3 + 2 * 3 = 9, \delta_4 = 1$

$r = 4, \delta_1 = 1, \delta_2 = 2 + r = 2 + 4 = 6, \delta_3 = 3 + 2r = 3 + 2 * 4 = 11, \delta_4 = 1$

Optimal primal value can also be searched using the equality conditions of both constraint and objective functions as follows.

$\frac{U_1}{\delta_1} = \frac{f_1}{\delta_1} = \frac{f_1}{1} = f_1$. Then we get that $U_1 = f_1 = 2x_1^{-2}x_2^{-3}$. From the constraint part we have equality conditions that:

$$\frac{U_2}{\delta_2} = \frac{U_3}{\delta_3} = \frac{U_2 + U_3}{\delta_2 + \delta_3} = \frac{1}{\delta_2 + \delta_3}$$

$$\frac{U_2}{\delta_2} = \frac{1}{\delta_2 + \delta_3}$$

$$x_1 = \frac{\delta_2}{\delta_2 + \delta_3}$$

But also we can find the optimal primal variable two in the same way that:

$$U_2 = \frac{\delta_2}{\delta_2 + \delta_3}$$

$$x_2 = \frac{\delta_3}{\delta_2 + \delta_3}$$

From the constraint three we get that:

$$\frac{U_4}{\delta_4} = \frac{1}{\delta_4}$$

$$U_4 = 1$$

$$\frac{x_1^{-1}x_2^{-2}}{\varepsilon_1} = 1$$

$$\frac{1}{x_1x_2^2} = \varepsilon_1$$

$$x_1x_2^2 = \frac{1}{\varepsilon_1}$$

But we have $x_1 = \frac{\delta_2}{\delta_2+\delta_3}$and $x_2 = \frac{\delta_3}{\delta_2+\delta_3}$, then substitute as follows:

$$x_1x_2^2 = \frac{1}{\varepsilon_1}$$

$$x_1\left(\frac{\delta_3}{\delta_2 + \delta_3}\right)^2 = \frac{1}{\varepsilon_1}$$

$$x_1 = \frac{1}{\varepsilon_1} * \left(\frac{\delta_2 + \delta_3}{\delta_3}\right)^2$$

$$(\frac{\delta_2}{\delta_2 + \delta_3})x_2^2 = \frac{1}{\varepsilon_1}$$

$$x_2^2 = \frac{(\delta_2 + \delta_3)}{\delta_2 \varepsilon_1}$$

$$x_2 = \left(\frac{\delta_2 + \delta_3}{\delta_2 \varepsilon_1}\right)^{1/2}$$

Now by substituting the above given values, epsilon one and dual variable we can get optimal primal values and we compute the values of both primal and dual values of the objective function $f_2(x)$.

Table 3: shows the epsilon constraint method values for second objective function

Opti. dual variable				Opt. Primal variable			Opt. Primal
δ_1	δ_2	δ_3	δ_4	x_1	x_2	ε_1	$f_2(x)$
1.000	1.000002	2.000003	0.000001	0.3333	0.66666	6.75	60.790
1.000	1.0200	2.03000	0.01000	0.3342	0.6653	6.755	60.800
1.000	1.4000	2.60000	0.20000	0.3491	0.6482	6.800	60.2409
1.000	1.0400	2.06000	0.0200	0.3354	0.6645	6.750	60.600
1.000	1.2000	2.30000	0.10000	0.3430	0.6573	6.751	59.880
1.000	5.000	8.0000	2.0000	0.3838	0.6147	6.880	58.460
1.000	15.000	23.0000	7.0000	0.3956	0.6060	6.900	57.471

From the above tabe 3 we can conclude thatprimal and dual decision variables values are $\delta_1 = 1.000, \delta_2 = 15.00, \delta_3 = 23, \delta_4 = 7.000, x_1 = 0.3956, x_2 = 0.6060\ with\ f_1(x) = 57.4712.$

5.5. Neurosophic method

Multi-objective non-linear programming problem were solved by using Neutrosophic geometric programming technique. As the Neutrosophic optimization technique utilizes degrees of Truth-membership, Falsity-membership and Indeterminacy-membership functions, we made a study of correspondence among those membership functions to see its impact on optimization. Also, we made a comparative study of optimal solution between weighed sum, fuzzy geometric programming and Neutrosophic geometric programming technique. The developed algorithm has been illustrated by example.

Step 1: Solve the MOGPP problem (1.1) as a single objective non-linear problem p times for each problem by taking one of the objectives at a time andignoring the others. These solutions are known as ideal solutions. Let x_k be therespective optimal solution for the kth different objective and evaluate eachobjective value for all these kth optimal solution.

Step 2: From the result of step-1, determine the corresponding values forevery objective for each derived solution. With the values of all objectives at eachideal solution, pay-off matrix can be formulated as follows.

$$\begin{bmatrix} f_1(X_1) & f_2(X_1)\dots f_k(X_1) \\ f_1(X_2) & f_2(X_2)\dots f_k(X_2) \\ f_1(X_3) & f_2(X_3)\dots f_k(X_3) \\ . & . \quad . \\ . & . \quad . \\ . & . \quad . \quad . \\ f_1(X_k) & f_2(X_k)\dots f_k(X_k) \end{bmatrix}$$

Here $X = (X_1, X_2 \dots, X_P)$ are the ideal solution of the $f_1(X), f_2(X), f_3(X), \dots, f_k(X)$ respectively. Hence we can write lower and upper bounds of each objective functions by using the above pay-off matrix. $U_k = \max\{f_k(X_k), k = 1,2,3,\dots,p\}$and also lower bound can be written as $L_k = \min\{f_k(X_K), k = 1,2,3,\dots,p\}$. Generally, it can be written as the best value L_k (minimum value) and the worst value U_r (maximum value) of each objective function $f_k(X)$ such that $L_k \leq f_k(X) \leq U_k, k = 1,2,3,\dots,p$. This lower and upper baunds of each objective function helps us to find different types of membership functions such as of truth-membership, falsity-membership and indeterminacy-membership functions.

Step-3: Werepresent upper and lower bounds for indeterminacy and falsity membership of objective functions as follows:

$$U_k^{v} = U_K^{\mu} \ and \ L_k^{v} = L_k^{\mu} + t(U_K^{\mu} - L_k^{\mu})$$

$$L_k^{\sigma} = L_k^{\mu} \ and \ U_k^{\sigma} = L_k^{\mu} + s(\, U_K^{\mu} - L_k^{\mu})$$

From the above equation $s \ and \ t$ are predetermined real numbers betweewn zero and one.

Step-4. Define Truth-membership, Indeterminacy-membership, Falsitymembershipfunctions as follows by using the above computed values.

$$\mu_k(x) = \begin{cases} 1 & if\ f_k(x) \le L_k \\ \dfrac{U_k - f_k(x)}{U_k - L_k} & if L_k \le f_k(x) \le U_k \\ 0 & if\ f_k(x) > U_k \end{cases}$$

$$v_k(x) = \begin{cases} 1 & if\ f_k(x) \ge U_k^v \\ \dfrac{f_k(x) - L_k^v}{U_k^v - L_k^v} & if L_k \le f_k(x) \le U_k \\ 0 & if\ f_k(x) \le L_k^v \end{cases}$$

$$\sigma_k(x) = \begin{cases} 0 & if\ f_k(x) \ge U_k^\sigma \\ \dfrac{U_k^\sigma - f_k(x) - L_k^v}{U_k^\sigma - L_k^\sigma} & if L_k \le f_k(x) \le U_k \\ 1 & if\ f_k(x) \le L_k^\sigma \end{cases}$$

And we have that $0 \le \mu_k(f_k(x)) + v_k(f_k(x)) + \sigma_k(f_k(x)) \le 3, k = 1,2,3,\dots,p.$

Step 7. Now a Neutrosophic geometric programming technique for multi-objectivenon-linear programming problem with the linear Truth-membership, Falsity-membership and Indeterminacy functions can be written as:

$$\text{Maximize } (\mu_1 \left(f_1(x), \mu_2(f_2(x)), \mu_3(f_3(x)), \dots, \mu_k(f_k(x))\right))$$

$$\text{minimize}(v_1 \left(f_1(x), v_2(f_2(x)), v_3(f_3(x)), \dots, v_k(f_k(x))\right))$$

$$\text{Maximize}(\sigma_1 \left(f_1(x), \sigma_2(f_2(x)), \sigma_3(f_3(x)), \dots, \sigma_k(f_k(x))\right))$$

$$\text{subject to } g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \le 1, i = 1,2,\dots,m$$

$$x_j > 0, j = 1,2,\dots,n$$

Now by using the weighted sum method we can transform the above formulated membership functions into single objective geometric programming problems as follows:

$$\text{Min } g(x) = \sum_{k=1}^{2} w_k(v_k(f_1(x)) - \mu_k(f_1(x)) - \delta_k(f_1(x))$$

$$= w_1 v_1 + w_2 v_2 - (w_1\mu_1 + w_2\mu_2) - (w_1\delta_1 + w_2\delta_2)$$

(By using the above relations we got the following results)

$$= \left(1 + \frac{1}{1-t} + \frac{1}{s}\right)\sum_{k=1}^{p}\left(\frac{w_k f_k(x)}{U_k - L_k}\right)$$

$$\text{s.t. } x_1 + x_2 \le 1$$

$$x_i > 0. \forall i$$

Excluding the constant term, the above problem reduced to the followinggeometric programming problem.

Example 5.5. (Pintu et al.,2014).

$$\text{Min} f_1(x) = x_1^{-1} x_2^{-2}$$

$$\text{Min } f_2(x) = 2x_1^{-2} x_2^{-3}$$

$$\text{s.t. } x_1 + x_2 \le 1$$

$$x_i > 0. \forall i$$

Solution: Neutrosophic geometric programming technique has the same procedure with fuzzy method except its membership function that the fuzzy method only used true membership function. But NS method used three membership function such as true membership function, false membership function and indeterminacy membership function. First we should find the ideal solutions of each objective functions which are used to find the lower and upper bunds of the objective functions. It also used to determine the membership functions. Now we can divide the problem into two sub-problems.

Sub-Problem One:

$$\text{Min } f_1(x) = x_1^{-1} x_2^{-2}$$

$$\text{s.t.} x_1 + x_2 \le 1$$

$$x_i > 0. \forall i$$

This problem is already solved in the fuzzy Geometric programming problem method and we simply used the solution. This sub-problem one and sub-problem two are already solved in the fuzzy GP method above as:

$X_1 = (x_1, x_{2)=}(\frac{1}{3}, \frac{2}{3})$and $X_2 = (x_1, x_{2)=}(\frac{2}{5}, \frac{3}{5})$ and the functional values of the objective functions are $f_1(\mathrm{X}_1) = 6.75,\ \mathrm{f}_1(\mathrm{X}_2) = 6.94,\ \mathrm{f}_2(\mathrm{X}_1) = 60.78$ and $\mathrm{f}_2(\mathrm{X}_2) = 57.87$

Now we can construct the pay-off matrix by using the ideal solutions of the two objective functions which used to know the lower and upper bound of each objective functions.

$\begin{bmatrix} f_1(X_1)f_2(X_1) \\ f_1(X_2)f_2(X_2) \end{bmatrix} = \begin{bmatrix} 6.75 & 60.75 \\ 6.94 & 57.87 \end{bmatrix}$, From the pay-off matrix the lower and upper bound can be written as follows. $L_1 = 6.75 \leq f_1(X_1) \leq 6.94 = U_1$

$L_2 = 57.87 \leq f_2(X_1) \leq 60.75 = U_2$ the above lower and upper bounds of each objective function helps us to formulate the membership functions of the two objective functions as follows.

$$\mu_1(x) = \begin{cases} 1 & if f_1(x) \leq 6.75 \\ \dfrac{6.94 - f_1(x)}{0.19}, & if\ 6.75 \leq f_1(x) \leq 6.94 \\ 0, if\ f_1(x)) > 6.94 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & if f_2(x) \leq 57.87 \\ \dfrac{60.75 - f_2(x)}{2.88}, & if\ 57.87 \leq f_2(x) \leq 60.75 \\ 0, if\ f_2(x)) > 60.75 \end{cases}$$

$$V_1(f_1(x)) = \begin{cases} 1 & if\ f_1(x) \geq U_1^V \\ \dfrac{f_1(x) - L_1^V}{U_1^V - L_1^V}, & if\ L_1^V \leq f_1(x) \leq U_1^V \\ 0 & if\ f_1(x) \geq L_1^V \end{cases}$$

$$V_2(f_1(x)) = \begin{cases} 1 & if\ f_2(x) \geq U_2^V \\ \dfrac{f_2(x) - L_2^V}{U_2^V - L_2^V}, & if\ L_2^V \leq f_2(x) \leq U_2^V \\ 0 & if\ f_2(x) \geq L_2^V \end{cases}$$

$$\delta_1(f_1(x)) = \begin{cases} 1 & if\ f_1(x) \leq L_1^\delta \\ \dfrac{U_1^\delta - f_1(x)}{U_1^\delta - L_1^\delta}, & if\ L_1^\delta \leq f_1(x) \leq U_1^\delta \\ 0 & if\ f_1(x) > U_1^\delta \end{cases}$$

$$\delta_2(f_2(x)) = \begin{cases} 1 & if\ f_2(x) \leq L_2^\delta \\ \dfrac{U_2^\delta - f_2(x)}{U_2^\delta - L_2^\delta}, & if\ L_2^\delta \leq f_2(x) \leq U_2^\delta \\ 0 & if\ f_2(x) > U_2^\delta \end{cases}$$

Now by using the above membership functions with the help of Max-addition operator we can transform multi objective geometric programming problems into single objective geometric programming problems. Where $\delta_k\big(f_k(x)\big)$, $V_k\big(f_k(x)\big) and \mu_k\big(f_1(x)\big)$ are indeterminacy, false and true membership functions respectively and the sum of all this membership functions are between zero and three.We use the weighted sum method used to transform to single objective geometric programming problem.

$$\text{Min } g(x) = \sum_{k=1}^{2}\big(v_k\big(f_1(x)\big) - \mu_k\big(f_1(x)\big) - \delta_k\big(f_1(x)\big)$$

$$= v_1 + v_2 - (\mu_1 + \mu_2) - (\delta_1 + \delta_2)$$

$$\text{s.t. } x_1 + x_2 \le 1$$

$$x_i > 0. \forall i$$

(i). $\boldsymbol{v_1 + v_2} = \left(1 - \frac{1}{1-t}\mu_1\right) + \left(1 - \frac{1}{1-t}\mu_2\right)$

$$= \left(1 - \frac{1}{1-t}\left(\frac{6.94 - f_1(x)}{0.19}\right)\right) + \left(1 - \frac{1}{1-t}\left(\frac{60.75 - f_2(x)}{2.88}\right)\right)$$

$$= -\left(\frac{6.94}{0.19(1-t)} + \frac{60.75}{2.88(1-t)}\right) + \left(\frac{f_1(x)}{0.19(1-t)} + \frac{f_2(x)}{2.88(1-t)}\right.$$

$$= k_1 + \left(\frac{f_1(x)}{0.19(1-t)} + \frac{f_2(x)}{2.88(1-t)}\right)$$

$$= k_1 + \left(\frac{5.269 f_1(x)}{(1-t)} + \frac{0.699 f_2(x)}{(1-t)}\right.$$

(ii). $\boldsymbol{\mu_1 + \mu_2} = \left(\frac{6.94 - f_1(x)}{0.19}\right) + \left(\frac{60.75 - f_2(x)}{2.88}\right)$

$$= \frac{6.94}{0.19} + \left(\frac{60.75}{2.88} - (5.269 f_1(x) + 0.699 f_2(x))\right)$$

$$= k_2 - (5.269 f_1(x) + 0.699 f_2(x))$$

$\boldsymbol{iii}$). $\boldsymbol{\delta_1 + \delta_2} = \left(\frac{1}{s}\mu_1 - \left(\frac{1-s}{s}\right)\right) + \left(\frac{1}{s}\mu_2 - \left(\frac{1-s}{s}\right)\right)$

$$= \left(\frac{1}{s}\left(\frac{6.94 - f_1(x)}{0.19}\right) - \left(\frac{1-s}{s}\right)\right) + \left(\frac{1}{s}\left(\frac{60.75 - f_2(x)}{2.88}\right) - \left(\frac{1-s}{s}\right)\right)$$

$$= \left(\frac{6.94}{0.19s} - \frac{f_1(x)}{0.19s} - \frac{(1-s)}{s}\right) + \frac{60.75}{2.88s} - \frac{f_2(x)}{2.88s} - \frac{(1-s)}{s}$$

$$= \left(\frac{6.94}{0.19s} + \frac{60.75}{2.88s} - \frac{(1-s)}{s} - \frac{(1-s)}{s}\right) - \left(\frac{5.269 f_1(x)}{s} + \frac{0.699 * f_2(x)}{s}\right.$$

$$= k_3 - \left(\frac{5.269 f_1(x)}{s} + \frac{0.699 * f_2(x)}{s}\right.$$

Considering equal importance of all truth, falsity and indeterminacy membership functions and we can assume $s = t$ which are pre-determined real numbers lies beween zero and one.

From (i), (ii) and (iii) we get that:

$$\text{Min } g(x) = \sum_{k=1}^{2}(v_k(f_k(x)) - \mu_k(f_k(x)) - \delta_k(f_k(x)) = v_1 + v_2 - (\mu_1 + \mu_2) - (\delta_1 + \delta_2)$$

$$= k_1 + (\frac{5.269f_1(x)}{(1-t)} + \frac{0.699f_2(x)}{(1-t)} - (k_2 - (5.269f_1(x) + 0.699f_2(x)) \quad - (k_3$$

$$- (\frac{5.269f_1(x)}{s} + \frac{0.699 * f_2(x)}{s})$$

$$= k + \frac{5.269f_1(x)}{(1-t)} + 5.269f_1(x) + \frac{5.269f_1(x)}{s} + \frac{0.699f_2(x)}{(1-t)} + 0.699f_2(x)$$

$$+ \frac{0.699 * f_2(x)}{s}$$

$$= \mathrm{k} + [5.269f_1(x)\left(\frac{1}{(1-t)} + 1 + \frac{1}{s}\right) + 0.699f_2(x)(1 + \frac{1}{1-t} + \frac{1}{s})$$

$$= k + \left(\frac{1}{(1-t)} + 1 + \frac{1}{s}\right)(5.269f_1(x) + 0.699f_2(x))$$

Therefore the following is the result of the above summation of membership's function we cancel out the constant term because it is maximization.

$$\text{Min } g(x) = \left(1 + \frac{1}{1-t} + \frac{1}{s}\right)(5.269f_1(x) + 0.699f_2(x))$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$x_i > 0. \forall i$$

Solution: let $s = t = 0.5$ then we substitute above then $1 + \frac{1}{1-t} + \frac{1}{s} = 1 + \frac{1}{1-0.5} + \frac{1}{0.5} = 1 + \frac{1}{0.5} + \frac{1}{0.5} = \frac{2}{0.5} = 4$ then we substitute above

$$\text{Min } g(x) = 21.076x_1^{-1}x_2^{-2} + 5.592x_1^{-2}x_2^{-3}$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$x_i > 0. \forall i$$

Now we can solve this problem using primal-dual relation using AM-GM inequality as follows.

$$\text{Min } Z(x) = \left(\frac{U_1}{\delta_1}\right)^{\delta_1}\left(\frac{U_2}{\delta_2}\right)^{\delta_2}\left(\frac{U_3}{\delta_3}\right)^{\delta_3}\left(\frac{U_1}{4}\right)^{\delta_4}\lambda^{\lambda}$$

$$= \left(\frac{22.076x_1^{-1}x_2^{-2}}{\delta_1}\right)^{\delta_1}\left(\frac{3.057132\,x_1^{-2}x_2^{-3}}{\delta_2}\right)^{\delta_2}\left(\frac{x_1}{\delta_3}\right)^{\delta_3}\left(\frac{x_2}{\delta_4}\right)^{\delta_4}\lambda^{\lambda}$$

$$= \left(\frac{22.076}{\delta_1}\right)^{\delta_1}\left(\frac{3.057132}{1-\delta_1}\right)^{1-\delta_1}\left(\frac{1}{1-\delta_1}\right)^{2-\delta_1}\left(\frac{1}{1-\delta_1}\right)^{3-\delta_1}(5-2\delta_1)^{(5-2\delta_1)}$$

$$s.t.\begin{cases} -\delta_1 - 2\delta_2 + \delta_3 = 0 \\ -2\delta_1 - 3\delta_2 + \delta_4 = 0 \\ \delta_1 + \delta_2 = 1 \\ \delta_3 + \delta_4 = \boldsymbol{\lambda} \end{cases}$$

But we have that from this system we can understand that the DD =4-2-1=1 which implies that the system has many solution. We have the optimal dual variable and let us express each dual variable in terms of one variable and let $\delta_1 = r, \delta_2 = 1 - \delta_1, \delta_3 = 2 - \delta_1$ and $\delta_4 = 3 - \delta_1$ we should know the domain of the dual variable $\delta_i > 0$, $\delta_1 = r > 0, \delta_2 = 1 - \delta_1 > 0, \delta_3 = 2 - \delta_1 > 0$ *and* $\delta_4 = 3 - \delta_1 > 0$.

Therefore the optimal dual variable can be obtained by using differentiation as follows:

$$\frac{d(Z(\delta_1))}{d(\delta_1)} = 0$$

$$\frac{d(\left(\frac{22.076}{\delta_1}\right)^{\delta_1}\left(\frac{3.057132}{1-\delta_1}\right)^{1-\delta_1}\left(\frac{1}{1-\delta_1}\right)^{2-\delta_1}\left(\frac{1}{1-\delta_1}\right)^{3-\delta_1}(5-2\delta_1)^{(5-2\delta_1)})}{d(\delta_1)} = 0$$

Therefore we can compute the dual variable as:

$$22.046(1-\delta_1)(2-\delta_1)(3-\delta_1) = 3.057132\delta_1(5-2\delta_1)^2$$

From above optimal values of the problem above solved by Neutrosophic method is both primal and dual variables and the corresponding optimal functional values withdual and primal variables are $\delta_1 = 0.6260958, \delta_2 = 0.37390042, \delta_3 = 1.3739042$ *and* $\delta_4 = 2.3739042$ and primal values are $x_1 = 0.366588$ and $x_2 = 0.633411$ with the corresponding objectives functional value as $f_1(x_1, x_2) = 58.56211$ and $f_2(x_1, x_2) = 6.799086$.

We can concluded the results of the above solution methods such as weighted sum, fuzzy geometric programming, epsilon constrained and neutrosophic geometric programming methods which has different optimal values but we can compare the solution obtained from the table given below as follows.

Table 4: Comparison of optimal solutions obtained by FGP, NGP and WS methods

NO.	Optimization Techniques	Optimal decision Variables	Optimal Objective Functions
		x_1 and x_2	$f_1(x_1, x_2)$ and $f_2(x_1, x_2)$
1	WS	$0.392492,\ 0.60751$	$57.904,\ 6.903$
2	FGP	$0.360836, 0.6391634$	$58.82652,\ 6.783684$
3	NGP	$0.366588{,}0.633411$	$58.56211, 6.799086,$

From the above Tabe 4 we can concluded that weighted sum method due to the given example result is the best method. The results of WS is $f_1(x) = 57.904\ and\ f_2(x) = 6.903\ with\ x_1 = 0.392492\ and\ x_2 = 0.60751$. NGP is the second best method that obtained optimal objective functional value as $f_1(x) = 58.56211\ and\ f_2(x) = 6.799086\ with\ x_1 = 0.366588\ and\ x_2 = 0.633411$. The third method was FGP which had $f_1(x) = 58.82652,\ and\ f_2(x) = 6.783684\ with\ x_1 = 0.360836\ and\ x_2 = 0.6391634$ is same as the solution obtained by fuzzy programming method. However the weighted sum method gives a set of solution where the decision maker has a choice to choose his/her solution according to their choice. But fuzzy programming and neutrosophic methods gave only one solution.

6. Application of Multi-Objective Geometric Programming Problems

Multi-objective geometric programming optimization has many real life applications. Applications of geometric programming span many of the classical subsets of non-linear programming. Geometric programming theory also suggests very powerful computational techniques which make its study an interesting one for anyone concerned with non-linear optimization. Specifically, geometric programming is designed to minimize constrained generalized posynomials where the posynomial coefficients are required to be positive and the constraints (also posynomials) are bounded above by unity. These special polynomial functions, which are called posynomials.

Many engineering design problems as well as problems from business and economics can be modeled as constrained posynomials and solved using geometric programming approach. The geometric programming solution technique for these constrained posynomial problems consist of defining and solving an associated problem called the dual program. The optimal solution to the original problem (called the primal program) is then easily computed from the optimal solution of the dual program using the relationship provided by the duality theory. Optimization is performed on the dual program rather than the primal because the dual is always a concave maximization problem constrained by linear constraints.

Gravel box design

Here we have taken gravel box design problem. A total of 800 cubic meters of gravel is to be ferried across a river on a barrage. A box (with an open top) is to be built for this purpose. The transport cost per round trip of barrage of box is .05 birr, the costs of materials of the ends of the box are 20 birr per sqare meter. And other two sides and bottom are made from available scrap materials. Find the dimensions of the box that is to be built for this purpose to minimize the transport cost and material cost? Let length = x_1 m, width =x_2m , height = x_3 m. The area of the ends of the gravel box = $x_2 * x_3$ m^2. Area of the sides = $x_1 * x_3\ m^2$. Area of the bottom = $x_1 * x_2\ m^2$. The volume of the gravel box=$x_1 * x_2 * x_3\ m^3$. Let us assume that the length $L = x_1$ meter, Width $W = x_2$ meter and Height $H = x_3$ meter. The area of the end of the gravel box is $W * H = x_2 * x_3$ metere square, the area of the bottom of the gravel box is $L * W = x_1 * x_2$ square meter, the volume of the gravelbox is $L * W * H = x_1 *$

$x_2 * x_3$ meter cubic. The cost functions are transport cost and material cost. Let $f_1(x)$ be total cost which is equal to the sum of transport cost and material cost:

i.e. Total cost $(f_1(x))$ = Transport cost + restricted Material cost

$$= \frac{40}{x_1 x_2 x_3} + (2A_2)$$

$$= \frac{40}{x_1 x_2 x_3} + 2 * 20 * (x_2 * x_3),$$

$$= \frac{40}{x_1 x_2 x_3} + 40 x_2 x_3$$

We only restricted on the end side of the box.

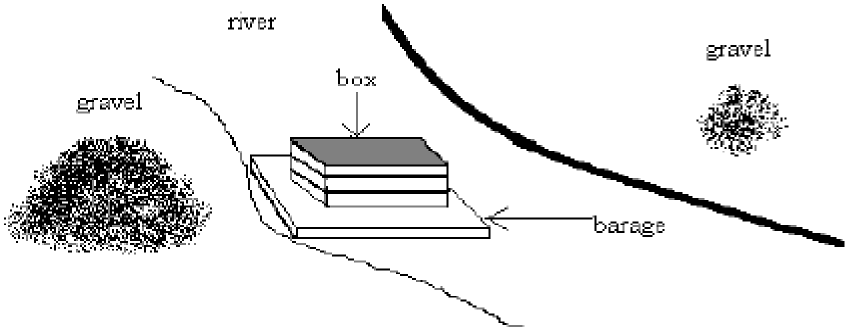


Figure 6 gravel transportation from one working area to other working area

We can apply MOGPP on gravel box design. We want to transport a massive amount of gravel from one working area to other working area across a river using a gravel box. Here we need to optimize the dimention of the box and we need to minimize transportion of material and trip. Now from the above information we can find the following values. From the above box we have two times the end of the box area, two times the side of the box area and one bottom area because the top is opened. Here we also find the maximum volume of $x_{1*} x_2 * x_3$ meter cubic to finish the transportation of the gravel from area one to area two across the river. We minimize the cost of transportation and cost of material used to build the gravel box. Let 1 trip = 0.05 birr, but we have 800 meter cubic gravel then we can get the number of trips as $\frac{800}{x_{1*} x_2 * x_3}$. To finish transporting the gravel, we should know the total cost of ransporting, total material cost which is used to bulid gravel box as follows.

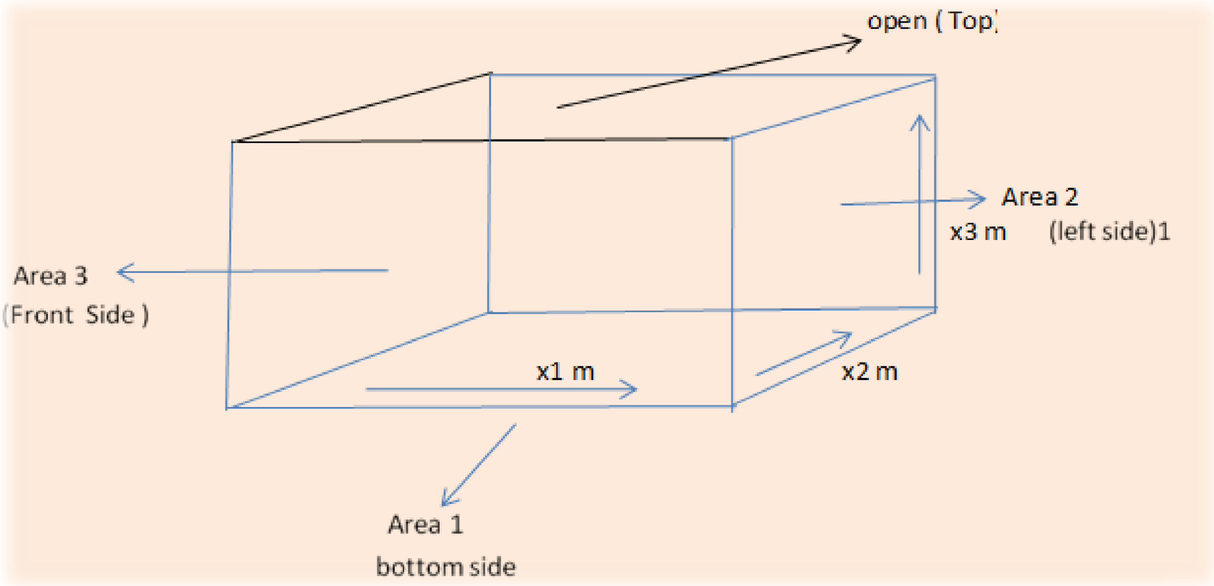


Figure 7: Designing gravel box

Since we are limited to a certain square meter for the material not need to be exceed, then we can change it into constrained geometric programming problem. We only restrict the required part of the box. Side, bottom, end or other part of the box. Ethir one part two part of three part.To make the above problem multi-objective let us also minimize the number of trips. Then we can write the above multi objective geometric programming problem as follows.

$$\text{Min } f_2(x) = \frac{40}{x_1x_2x_3} + 40x_2x_3$$

$$\text{Min } f_1(x) = \frac{800}{x_1x_2x_3}$$

$$\text{s.t. } x_1x_2 + 2x_1x_3 \leq 4$$

$$x_i > 0. \forall i$$

Now let us solve the above multi-objective geometric problem by weighted sum method as follows.

$$\text{Min} f_1(x) = 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_2x_3$$

$$\text{Min } f_2(x) = 800x_1^{-1}x_2^{-1}x_3^{-1}$$

$$s.t. \;\; x_1x_2 + 2x_1x_3 \leq 4$$

$$x_i > 0. \forall i$$

Solution: The weighting problem corresponding to the above problem is:

$$\text{Min f(x)} = \sum_{k=1}^{p} w_k\, f_k(x) = w_1 f_1 + w_2 f_2$$

$$= \ w_1[40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_2x_3] + w_2[800x_1^{-1}x_2^{-1}x_3^{-1}]$$

$$= [40w_1 + 800w_2]\ x_1^{-1}x_2^{-1}x_3^{-1} + w_1 40x_2x_3$$

$$\text{s.t.}\quad \frac{1}{4}x_1x_2 + \frac{1}{2}x_1x_3 \leq 1$$

$$x_i > 0. \forall i, w_i > 0\ , i = 1{,}2\ and\ w_1 + w_2 = 1$$

we can identify the degree of difficulty i.e. DD=4-3-1=0, this implies that the problem has unique solution means it has the same numbers of equations and decision variable. Now to solve the problem we use AM-GM inequality relation.

$$\text{Min} f(x) \geq \left(\frac{U_1}{\delta_1}\right)^{\delta_1} \left(\frac{U_2}{\delta_2}\right)^{\delta_2} \left(\frac{U_3}{\delta_3}\right)^{\delta_3} \left(\frac{U_1}{4}\right)^{\delta_4} \lambda^{\lambda}$$

$$= \left(\frac{[40w_1 + 800w_2]x_1^{-1}x_2^{-1}x_3^{-1}}{\delta_1}\right)^{\delta_1} \left(\frac{w_1 40x_2x_3}{\delta_2}\right)^{\delta_2} \left(\frac{\frac{1}{4}x_1x_2}{\delta_3}\right)^{\delta_3} \left(\frac{\frac{1}{2}x_1x_3}{\delta_4}\right)^{\delta_4} (\delta_3 + \delta_4)^{(\delta_3+\delta_4)}$$

$$\text{s.t.} \begin{cases} -\delta_1 + \delta_3 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ \quad \delta_1 + \delta_2 = 1 \\ \quad \delta_3 + \delta_4 = \lambda \end{cases}$$

Then we can compute the values of the dual variables as$\delta_1 = \frac{2}{3}$, $\delta_1 = \frac{2}{3}$, $\delta_{12} = \frac{2}{3}$, $\delta_3 = \frac{2}{3}$and $\delta_4 = \frac{2}{3}$. Then we can compute the decision variables by using equality conditions from above as follows. $\frac{U_1}{\delta_1} = \frac{U_2}{\delta_{12}}$, taken from the objective function and $\frac{U_3}{\delta_3} = \frac{U_4}{\delta_4}$ are taken from the constrained part. Then we can equate as:

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_{12}} = \frac{U_1 + U_2}{\delta_1 + \delta_2} = \frac{f}{1}$$

$$\frac{[40w_1 + 800w_2]x_1^{-1}x_2^{-1}x_3^{-1}}{\delta_1} = \frac{w_1 40x_2x_3}{\delta_2}$$

$$[40w_1 + 800w_2]\delta_2 x_1^{-1}x_2^{-1}x_3^{-1} = w_1\delta_1 40x_2x_3$$

$$(x_2x_3)(x_1x_2x_3) = \frac{[40w_1 + 800w_2]\delta_2}{40w_1\delta_1}$$

But also from the constrained part we have

$$\frac{U_3}{\delta_3} = \frac{U_4}{\delta_4} = \frac{U_3 + U_4}{\delta_3 + \delta_4} = \frac{1}{\lambda} = \frac{3}{2}$$

$$\frac{\frac{1}{4}x_1x_2}{\delta_3} = \frac{3}{2} \text{ and } \frac{\frac{1}{2}x_1x_3}{\delta_4} = \frac{3}{2}$$

$$x_1x_2 = 2 \text{ and } x_1x_3 = 1$$

Now from above, we get that$x_3 = (\frac{w_1+20w_2}{8w_1})^{\frac{1}{3}}$, $x_2 = 2x_3$.Weighting problem for a given summarize the non-inferior solutions by varying the weights in the interval [0,1] by the following table.

Table 5: Optimal solutions obtained by the weighted sum method

		Optimal Primal Variables			Opt. Pri. of Obj. Fun		
weights w_1, w_2	Opti. Dual Var.	x_1	x_2	x_3	$f_1(x)$	$f_2(x)$	Sum
0.1&0.9	$\delta_1 = 0.667$	0.3535	5.6566	2.8283	647.01	141.45	788.46
0.2&0.8	$\delta_2 = 0.333$	0.4622	4.3267	2.1633	383.64	184.92	568.56
0.3 &0.7	$\delta_3 = 0.667$	0.5516	3.6258	1.8129	273.96	220.64	494.60
0.4 &0.6	$\delta_4 = 0.333$	0.6366	3.1413	1.5706	210.08	254.71	464.79
0.5 & 0.5		**0.7249**	**2.7589**	**1.3794**	**166.72**	**289.99**	**456.71**
0.6 &0.4		0.8233	2.4291	1.2145	134.47	329.37	463.84
0.7 &0.3		0.9419	2.1232	1.0616	109.00	376.81	485.81
0.8 &0.2		1.1006	1.8171	0.9085	88.04	440.30	528.34
0.9 &0.1		1.3540	1.4770	0.7385	70.71	541.67	612.38

From Table 5, we have different optimal solutions for different weights of the problem by weighted-sum method. First objective gives better optimal result when w_1 increases. Similarly second objective gives better optimal result when w_2 increases and the number of trips to transport the given gravel is 167 and the total cost is 290 with weighted scalar constant $w_1 = 0.5$ and $w_2 = 0.5$.

7. SUMMARY, CONCLUSION AND RECOMMENDATION

7.1.Summmary

Optimization is the act of obtaining the best result under given circumstances. Most optimization problems naturally have several conflicting objectives to be satisfied, for such types of problems the single objective approach appears very limited for the nature of design problems. Because of such class of problems, we introduce a multi-objective optimization. It is a class of problems with solutions that can be evaluated along two or more incomparable or conflicting objectives. To solve a MOOP we used the WS, ε-constraint, fuzzy and neutrosophic methods.

The WS can be done by scalarizing the set of objectives into a single objective by adding each objective pre-multiplied by a user supplied weight under a given constraints and the way of choosing the weights are very sensitive. Weight of an objective is chosen in proportion to the relative importance of the objective. In the ε-constraint method, we select one objective functions out of k to optimize and converting the remaining objectives to constraints putting a target values each of them. Because of this the number of constraints increases and it takes high computational time to find the optimal solution.

7.2. Conclusion

Multi-objective geoemetric programming problems can be solved by using WS, ε-constraint, fuzzy and neurosophic methods. In multi-objective optimization problems, there is no unique solution exists but a set of mathematically equally good solutions can be identified. One of the most widely used methods for solving multi-objective optimization problems is to transform a multiple objective optimization problem into a series of single objective optimization problems.

Getting of a suitable compromise solution corresponding to a multi-objective geoemetric programming problem is a difficult task due to conflict between various objectives and goals. In this project we have used the weighted sum method which obtained best result when comparing the other methods. By using weighted method we can solve a multi-objective GPP as a vector minimum problem. The weighting method is the simplest multi-objective optimization which has been widely applied to find optimal solution of multi-objective function within the convex objective space.

We use ε-constraint method to find the set of primal and dual solutions of the multi-objective functions with given constraints. The corresponding fuzzy programming techniques have been applied to find the optimal values of the functions. From the computation it has been observed that the optimal solution obtained by primal-dual techniques matches with their counterpart solution due to fuzzy programming method.

Weighted sum method is that it can be difficult to discern between setting weights to compensate for differences in objective function and setting weights to indicate the **relative importance** of an objective as it is done with the rating methods. Since weighted sum approach involves a weight for each objective, it is sometimes difficult to achieve the task due to the objectives importance. An ε-constraint method applicable to either convex or non-convex problems.The ε-values has to be chosen carefully so that it is within the minimum or maximum values of the individual objective function. In general, multi-objective optimization requires more computational effort than single objective optimization.

7.3. Recommendations

Solving multi-objective geometric programming problems using the weighted sum, ε-constraint, fuzzy and neutrosophic methods, the following basic recommendations are suggested:

- Multi-objective optimization problems were solved by converting into single objective optimization problems by using different methods.
- From the four methods, the WS is best in saving computational time. The illustrative examples also show that it requires more computational time.
- Multi objective geoemteric programming problems that are solved by weighted sum method may not solve by fuzzy, epsilon and neutrosophic methods unless the degree of difficulty is changed to either zero or positive by further investigation.
- This project deals with solving MOGPP with positive degree of difficulty. As a first open problem, we may generalize this type of GPP problems by solving them for degree of difficulties starting from (-1), (-2), (-3),… and so on.

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