FUNCTIONAL IDENTITIES ON SEMIPRIME RINGS WITH MULTIPLICATIVE (GENERALIZED)-DERIVATIONS

MSc PROJECT

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MARCH 2019

HARAMAYA UNIVERSITY, HARAMAYA

Functional Identities on Semiprime Rings with Multiplicative (Generalized)-Derivations

A Project Submitted to Department of Mathematics

Postgraduate Program Directorate

HARAMAYA UNIVERSITY

In Partial Fulfillment of the Requirements for the degree of

MASTER OF SCIENCE IN MATHEMATICS

(ALGEBRA)

HAILU FIKIRE FOKORO

MARCH 2019

Haramaya University, Haramaya

HARAMAYA UNIVERSITY

Postgraduate Program Directorate

I here by certify that I have read and evaluated this Project titled *'Functional identities on semiprime Rings with multiplicative (generalized)-derivations'* prepared under my guidance by Hailu Fikire. All feedback given to the student has been incorporated in the project. Therefore, I recommend that it be submitted as fulfilling the project requirement.

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As member of the board of Examiners of the MSc Project Open Defense Examination, we certify that we have read and evaluated the Project prepared by Hailu Fikire and examined the candidate. We recommend that the Project be accepted as fulfilling the Project requirement for the degree of Master of Science in Mathematics (Algebra).

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Chairperson Signature Date

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Internal Examiner Signature Date

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External Examiner Signature Date

DEDICATION

I dedicate this Project to my family members for their love and care in the success of my life and memorable and valuable encouragements in my academic career while I was doing this study.

STATEMENT OF THE AUTHOR

By my signature below, I declare that this project is my own work. I have followed all ethical and technical principles of scholarship in the preparation, and compilation of this project. Any scholarly matter that is included in the project has been given recognition through citation. This project is submitted in partial fulfillment of the requirements for Master of Science in Mathematics at Haramaya University. The project is deposited in the Haramaya University Library and is made available to borrowers under the rules of the Library. I solemnly declare that this project has not been submitted to any other institution anywhere for the award of any academic degree, diploma or certificate.

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BIOGRAPHICAL SKETCH OF THE AUTHOR

The author was born on September 14, 1988 G.C in Morsito Town which is located18 km from Hossana, Hadiya Zone, SNNPR. He attended his elementary education at Morsito Primary School. He completed his high School education at Wachemo Secondary and Praparatory Shool in 2004 G.C.

Up on successful completion of high School study, he joined Hawassa College of Teacher Education in September, 2005 G.C and graduate in 2007 G.C with Mathematics and physical science diploma and he joined Hawassa University in 2009 G.C and graduated in August, 2014 G.C with B.Sc degree in Applied Mathematics. Then he was employed by Hadiya Zone, Misha Woreda Education Bureau. He served the Woreda for 9 years until he joined Postgraduate Program at Haramaya University, College of Natural and Computational Sciences, Departement of Mathematics in 2016 G.C to pursue his MSc degree in Mathematics with specialization in Algebra.

ACKNOWLEDGEMENT

First of all I would like to thanks my almighty Lord, for giving this grace and blessing, starting up to final write up of this project. My profound sincere appreciation and thanks go to my course instructors and my project major Advisors **Dr. Phool Miyan** and Co-advisor **Dr. Seleshi Demie** who had been kindly and extensively advises me day to day suggesting the problems tirelessly, giving me constructive comments and make improvement on this project.

I wish to extend my sincere thanks to all my family and my beloved wife Mishame and my family, who have been constant source of encouragement and support. Finally, I would like to express my heartfelt thanks to Misha Woreda Education office and Haramaya University Mathematics department who gave me encouragement and helped me during this study.

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Functional Identities on Semiprime Rings with Multiplicative (Generalized)-Derivations

ABSTRACT

The main purpose of this project was to study some functional identities on semiprime rings with multiplicative (generalized)-derivations. This is because, which facilitate semiprime ring is essential in multiplicative (generalized)-derivation. The important preliminary concepts, examples, lemmas and theorems were presented to make the concept of functional identities on semiprime rings with multiplicative (generalized)-derivations more clear. In order to study functional identities on semiprime rings with multiplicative (generalized)-derivations used many methods such as commuting map, derivation, multiplicative derivation, left ideal, multiplicative left centralizer and multiplicative (generalized)-derivation were presented. The results show that semiprime ring is essential in all theorems and multiplicative (generalized)-derivations is commuting map on left ideal. Finally, results of some examples shows that essentiality semiprimeness.

Key words: *Commuting map, Derivation, left ideal, multiplicative (generalized)-derivation multiplicative left centralizer, semiprime ring.*

1. INTRODUCTION

1.1. Background of the Study

The study of rings originated from the theory of polynomial rings and the theory of algebraic integers. A ring is an abelian group with first binary operation and with second binary operation that is associative, is distributive over the abelian group operation, and has an identity element. By extension from the integers, the abelian group operation is called addition and the second binary operation is called multiplication.

A ring R is called prime ring if for any $a, b \in R, aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ and is called semiprime if $aRa = \{0\}$ implies that $a = 0$.

Posner (1957) defined derivation in ring as follows An additive mapping $d: R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all x, y element of a ring R.

Following above definition, Bresar (1991) introduced the concept of generalized derivations in rings as follows:- An additive mapping $F: R \to R$ is called a generalized derivation if there exists a derivation d on a ring R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Thereafter Hvala (1998) has introduced the notion of generalized derivations in prime rings.

Motivated by above definitions, Daif (1991) introduced the concept of multiplicatve derivation as follows a mapping $d: R \to R$ such that $d(xy) = d(x)y + xd(y)$, but mapping is not additive for all x, y in R and was motivated by the work of (MartindaleIII,1969). Therefore complete description of those maps was given (Goldmann and semrl, 1996).

Based by above concepts, Daif and Bell (1992) proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, then I is a central ideal. In particular, if $I = R$ then R is commutative.

Ashraf and Rehman (2001) proved that if R is a prime ring with a nonzero ideal I of R and d is a derivation of R such that either $d(xy) - xy \in Z(R)$ for all $x, y \in I$ or $d(xy) + xy \in Z(R)$ for all $x, y \in I$, then R is commutative.

Recently Ashraf *et al.* (2007) have studied the situations when derivation d *is* replaced with a generalized derivation F. More precisely, they proved that a prime ring R must be commutative, if R satisfies any one of the following conditions for all $x, y \in I$:

$$(i)\ F(xy) - xy \in Z(R)$$

$$(ii)\ F(xy) + xy \in Z(R)$$

$$(iii)\ F(xy) - yx \in Z(R)$$

$$(iv)\ F(xy) + yx \in Z(R)$$

$$(v)\ F(x)F(y) + yx \in Z(R) \text{ and}$$

$$(vi)\ F(x)F(y) - yx \in Z(R)$$

where F is a generalized derivation of a ring R associated with a nonzero derivation d , $Z(R)$ is a center of a ring R and I is a nonzero two-sided ideal of R.

Daif and Sayiad (2007) Motivated by the definition of multiplicative derivation extended the notion of multiplicative derivation to multiplicative generalized derivation as follows a mapping $F: R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation from R into R.

Based by above definition, Dhara and Ali (2013) make a slight generalization of Daif derivation d as any map, defined that a mapping $F: R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivations if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is any map (not necessarily a derivation nor additive map) and they proved some functional identities on semiprime ring involving multiplicative (generalized)-derivations.

Thereafter Ali *et al.* (2015) proved some functional identities of semiprime ring involving multiplicative (generalized)- derivation in rings. In the same year Tiwari *et al.* (2015) proved the study of commutativity of semiprime rings and nature of mapping with multiplicative (generalized)-derivation in rings.

Recently Alhazmi (2016) proved some funcional identities involving two multiplicative (generalized)-derivation in prime rings. In the same year Deepak and Gurninder (2016) Proved some functional identities on semiprime ring with multiplicative (generalized)-derivation.

Very recently Camci and Aydin (2017) proved some functional identities with multiplicative (generalized)-derivation and multiplicative left centralizer.

In this project we discussed in detail different functional identities on semiprime rings with multiplicative (generalized)-derivations.

1.2. Statement of Problem

Daif (1991) introduced the concept of multiplicative derivation and Daif and Sayiad (2007) extended multiplicative derivation to multiplicative generalized derivations, thereafter multiplicative generalized derivation has been extended to multiplicative (generalized)-derivations by Dhara and Ali (2013).

To state and prove theorems that deal with essentiality of semiprime ring R in multiplicative (generalized)-derivation. In a connection with this many results have been stated proved in the last five years.

Dhara and Ali (2013) stated and proved some theorem about functional identities on semiprime ring R involving multiplicative (generalized)-derivations under the following conditions for all $x, y \in I$:

$$(i)\ \ F(xy)\ \pm xy = 0$$

$$(ii)\ \ F(xy) \pm yx = 0$$

$$(iii)\ \ F(xy) \pm yx \in Z(R)$$

$$(iv)\ \ F(xy) \pm xy \in Z(R)$$

$$(v)\ \ F(x)F(y) \pm xy = 0$$

$$(vi)\ \ F(x)F(y) \pm xy \in Z(R) \text{ and}$$

$$(vii)\ \ F(x)F(y) \pm yx \in Z(R).$$

where F is a multiplicative (generalized)-derivations of a ring R associated with any map d (not necessarily derivations) and I is a nonzero two- sided ideal of a ring R.

Camci and Aydin (2017) stated and proved some identities with multiplicative (generalized)-derivations F and multiplicative left centralizer H under the following conditions:

$$(i)\ \ F(x)F(y) \pm H(xy) = 0$$

$$(\,ii)\ \ F(xy) \pm H(yx) = 0$$

$$(iii)\ \ F(xy) \pm H(xy) = 0$$

$$(iv)\ \ F(xy) \pm H(xy) \in Z(R)$$

$$(v)\ \ F(xy) \pm H(yx) \in Z(R) \text{ and}$$

$$(vi)\ \ F(x)F(y) \pm H(xy) \in Z(R)$$

for all $x, y \in R$.

In this project we tried to:

(i) Provide all the necessary preliminary concepts and ideas on multiplicative (generalized)-derivations in semiprime rings.

(ii) Elaborate the ideas and concepts on functional identities on semiprime rings with multiplicative (generalized) derivations presented in Dhara and Ali (2013) and Camci and Aydin (2017).

(iii) Provide more detailed proof of the theorems stated and proved by Dhara and Ali (2013) and Camci and Aydin (2017).

1.3. Objective

Then main objective of this project was to study some functional identities on semiprime rings with multiplicative (generalized)-derivations.

The study explored the following specific objective

- To discuss the condition of multiplicative (generalized)-derivations to be commuting map on left ideal L.
- To discuss about essentiality of semiprimeness ring R in multiplicative (generalized)-derivations.
- To prove some theorems in the multiplicative (generalized)-derivations in semiprime rings with functional identities.

2. LITERATURE REVIEW

Ali *et al.* (2015) proved some functional identities of rings involving multiplicative (generalized)-derivations in rings and they studied the following identities:

$$(i)\ F(x)F(y) \pm [x,y] \in Z(R)$$

$$(ii)\ F(x)F(y) \pm x \circ y \in Z(R)$$

$$(iii)\ F([x,y]) \pm [x,y] \in Z(R)$$

$$(iv)\ F(x \circ y) \pm (x \circ y) \in Z(R)$$

$$(v)\ F([x,y]) \pm [F(x),y] \in Z(R)$$

$$(vi)\ F(x \circ y) \pm (F(x) \circ y) \in Z(R)\ \text{and}$$

$$(vii)\ [F(x),y] \pm [G(y),x] \in Z(R)$$

for all x, y in some appropriate subset of a ring R, where F and G are multiplicative (generalized)-derivations and $Z(R)$ is a center of a ring R.

Base on above results, Tiwari *et al*. (2015) proved the study of commutativity of semiprime rings and nature of mapping with multiplicative (generalized)-derivations in rings and they studied the following functional identities for all $x, y \in I$:

$$(i)\ G(xy) \pm F(x)F(y) \pm [a(x),y] = 0$$

$$(ii)\ G(xy) \pm F(x)F(y) \pm [x,a(y)] = 0$$

$$(iii)\ F(x \circ y) \pm d(x) \circ y = 0$$

$$(iv)\ G(xy) \pm [F(x),y] \pm yx = 0$$

$$(v)\ G(xy) \pm [x,F(y)] \pm yx = 0\ \text{and}$$

$$(vi)\ G(xy) \pm F(x)F(y) \pm x \circ y = 0$$

where I is a non-zero ideal in prime ring R and $a : R \to R$ is any mapping, G and F are two multiplicative (generalized)-derivations associated with the mappings d on R.

Motivated by above results, Alhazmi (2016) proved some functional identities involving two multiplicative (generalized)-derivations in rings and studied the commutativity of prime rings involving multiplicative (generalized)-derivations with annihilating conditions. More precisely, he studied the identities that $a(G(xy) + F(x)F(y) \pm yx) = 0$, for all $x, y \in I$, where I is a non zero ideal of R and F, G are multiplicative (generalized)-derivations of prime ring R.

Following above results, Deepak and Gurninder (2016) proved some identity of multiplicative (generalized)-derivations of rings in the setting of left ideals of rings and studied the following situations:

$$(i)\ \ F[x,y] \pm xy = 0$$

$$(ii)\ F[x,y] \pm yx = 0$$

$$(iii)\ F(x \circ y) \pm xy = 0$$

$$(iv)\ \ F(x \circ y) \pm yx = 0$$

$$(v)\ \ d(x)F(y) \pm xy = 0$$

$(vi)\ \ d(x)F(y) \pm yx = 0$ and

$$(vii)[F(x),y] \pm x \circ G(y) = 0$$

for all x, y in some appropriate subsets of R, where F, G are multiplicative (generalized)-derivations and d is any map (note necessarily a derivation nor additive).

3. METERIAL AND METHODS

This chapter outlines the methods and materials that was used to study functional identities on semiprime rings with multiplicative (generalized)-derivations and materials used in the study. Sources of relevant information collected on functional identities on semiprime rings with multiplicative (generalized)-derivations from internets, related updated journals, library, and web recorded subsequently. Specifically:

- The important preliminary concepts, definition, lemmas, semiprimeness, ideal, nilpotent ideal, examples and theorem was discussed to make the concept clear.
- The collected information was analyzed.
- Relevant journals were consulted to gather information about functional identities on semiprime rings with multiplicative (generalized)-derivations.
- Proved some theorem on functional identities on semiprime rings with multiplicative (generalized)-derivations to make the concept clear.

4. PRELIMINARY

This chapter contains basic definitions and examples in which we shall need for the development of the subject in the subsequent chapter of the present project.

4.1. Some ring theoretic notions

Definition 4.1.1. A non-empty set of R with two binary operation addition $(+)$ and multiplication $(\cdot)$, then $< R, +, \cdot >$ is said to be a ring R, if satisfying the following properties:

(1) $(R, +)$ is an Abelian group

(2) $(R, \ \cdot)$ is a semi group.

(3) Distributive laws hold

(i) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$, (Left distributive laws)

(ii) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a , b, c \in R$, (Right distributive laws)

Example 4.1.1. $< \mathbb{Z}, +, \cdot >$ is set of integer , $< \mathbb{Q}, +, \cdot >$ is set of a rational numbers and $< \mathbb{R}, +, \cdot >$ is a set of real numbers are examples of a ring R, under addition (+) and multiplication (·).

Definition 4.1.2. For any pair of elements x, y in a ring R, the multiplicative commutator denoted by $[x, y] = xy - yx$.

Properties commutator on rings

(i) $[xy, z] = x[y, z] + [x, z]y$ for all $x, y, z \in R$.
(ii) $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$.
(iii) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in R$.
(iv) $[x, y + z] = [x, y] + [x, z]$ for all $x, y , z \in R$.
(v) $[x, x] = 0$, since by definition 4.1.2. we get $xx - xx = 0$ for all $x \in R$.

Definition 4.1.3. A ring R is said to be prime if for any a , b $\in R, aRb = \{0\}$ implies either $a = 0$ or $b = 0$.

Definition 4.1.4. A ring R is said to be semiprime if for any $a \in R$ such that $aRa = \{0\}$ implies $a = 0$.

Definition 4.1.5. Let R be a ring, then center of a ring R is defined as the collection of all those members of R which commutes with every members of R and denoted by $Z(R)$. i.e. $Z(R) = \{a \in R | ax = xa \ \forall \ x \in R\}$.

Definition 4.1.6. A non-empty subset $I$of a ring R is called a right ideal of R if:

(i) $a, b \in I$ implies $a - b \in I$

(ii) $a \in I, r \in R$ implies $ar \in I$

and a left ideal of R if $ra \in I$ and I is called a two sided ideal of R if it is both left and right ideal of R.

Example 4.1.6. Even integer is both sided ideal of integer $\mathbb{Z}$.

Definition 4.1.7. Let R be a ring. An ideal A is said to be nilpotent if $A^n = 0$ for any natural number n, where n is the index of A.

Example 4.1.7. Zero ideal always nilpotent ideal as $0^1 = 0$.

4.2. Multiplicative (generalized)-derivation in rings

Definition 4.2.1. Let R be a ring and d is mapping from R to R, then d said to be derivation on R if :

(i) d is additive on ring R, that means $d(x + y) = d(x) + d(y)$ for all $x, y \in R$.

(ii) $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$.

Example 4.2.1. Let S be any ring, next, let $R = \left\{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in S\right\}$, and define the map $d: R \to R$ as follows: $d\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. Then d is derivation.

Verification:

(i) $d\left(\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}+\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}\right)=d\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}+d\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}$ that means d is additive on R.

(ii) $d\left(\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}\right)=d\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}+\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}d\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}$.

Then d is derivation on R.

Definition 4.2.2. An additive mapping $F: R \to R$ is called a generalized derivation of R, associated with a derivation $d: R \to R$, such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$.

Example 4.2.2. let S be any ring, next, let $R = \left\{\begin{pmatrix}0 & a\\0 & b\end{pmatrix} \mid a, b \in S\right\}$, and define the maps $d, F: R \to R$ as follows:

$d\begin{pmatrix}0 & a\\0 & b\end{pmatrix}=\begin{pmatrix}0 & a\\0 & 0\end{pmatrix}$ and $F\begin{pmatrix}0 & a\\0 & b\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & b\end{pmatrix}$. Then F is generalized derivation.

Verification:

(i) $F\left(\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}+\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}\right)=F\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}+F\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}$ that means F is additive on R.

(ii) $F\left(\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}\right)=F\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}+\begin{pmatrix}0 & a_1\\0 & b_1\end{pmatrix}d\begin{pmatrix}0 & a_2\\0 & b_2\end{pmatrix}$.

Then F is generalized derivation on R.

Definition 4.2.3. A multiplicative derivation of R is a map $d: R \to R$ which satisfies $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$ of course these maps are not additive.

Example 4.2.3. let S be any ring, next, let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in S \right\}$, and define the maps $d: R \to R$ as follows: $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then d is multiplicative derivation of R.

Definition 4.2.4. Let $H: R \to R$ be a map. If $H(xy) = H(x)y$ holds for all $x, y \in R$, then H is called a multiplicative left centralizer.

Example 4.2.4. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers and the maps $H: R \to R$ defined by $H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda a & \lambda b \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix}$, where $\lambda \in \mathbb{Z}$, then H is multiplicative left centralizer.

Verification:

$$H\left(\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}\right) = H\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & \lambda c_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then H is multiplicative left centralizer.

Definition 4.2.5. Let S be a non-empty subset of R. A map $F: R \to R$ is called centralizing on S if $[F(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$.

Definition 4.2.6. A map $F: R \to R$ is called multiplicative generalized derivation of R, if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Example 4.2.6. Let S be any ring, next, let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in S \right\}$, and define the maps d and $F: R \to R$ as follows:

$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then F is not multiplicative generalized derivation of R.

Definition 4.2.7. A map $F: R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation associated to a map $d: R \to R$ if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is multiplicative derivation on R (not necessarily a derivation nor additive map).

Example 4.2.7. Let S be any ring, next, let $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in S \right\}$, and define the maps d and $F: R \to R$ as follows:

$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $F\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then F multiplicative (generalized)-derivation of R.

Verification

(i) $d(x + y) \neq d(x) + d(y)$

(ii) $F(x + y) \neq F(x) + F(y)$

(iii) $F(xy) = F(x)y + xd(y)$ for all $x, y, z \in S$.

Since

$$(i) \quad d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}\right) \neq d\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix} \ \forall x, y, z \in S$$

This means d is not additive

(ii) $$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}\right) \neq F\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix} + F\begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}$$ for all $x, y, z \in S$. Then F is not additive

$$(iii) \quad F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ and

$$F\begin{pmatrix}0 & x_1 & y_1\\ 0 & 0 & z_1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & x_2 & y_2\\ 0 & 0 & z_2\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}0 & x_1 & y_1\\ 0 & 0 & z_1\\ 0 & 0 & 0\end{pmatrix}d\begin{pmatrix}0 & x_2 & y_2\\ 0 & 0 & z_2\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$ for all

$x, y, z \in S$. Then F is multiplicative (generalized)-derivation.

5. Multiplicative (Generalized)-Derivation on Semiprime Rings satisfying some Functional Identities

Dhara and Ali (2013) proved some functional identities of semiprime rings involving multiplicative (generalized)-derivations. We deals with their results by incorporating further explanatory steps in their proof.

The following theorems (Theorem 5.1, Theorem 5.2, and Theorem 5.3) gives the concepts on commuting map on left ideal L of semiprime ring with multiplicative (generalized)-derivation.

Theorem 5.1 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R \to R$ be a multiplicative (generalized)-derivation associated with the map $d: R \to R$. If $F(xy) \pm xy = 0$ for all $x, y \in L$, then $Ld(L) = \{0\}$, $F(xy) = F(x)y$ for all $x, y \in L$ and F is commuting map on left ideal L.

Proof. By the first hypothesis, we have

$$F(xy) - xy = 0 \text{ for all } x, y \in L. \tag{5.1}$$

Replacing y with yz, $z \in L$ in (5.1) we get

$$F(xyz) - xyz = 0 \text{ for all } x, y, z \in L.$$

Since $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, it follows that,

$$0 = F(xy)z + xyd(z) - xyz = (F(xy) - xy)z + xyd(z) \text{ for all } x, y, z \in L. \tag{5.2}$$

Application of (5.1) yields that

$$xyd(z) = 0 \text{ for all } x, y, z \in L. \tag{5.3}$$

Replacing y with $d(z)ry$, in (5.3) where $r \in R$, we get

$$xd(z)ryd(z) = 0.$$

In particular $y = x$, it follows that

$$xd(z)Rxd(z) = \{0\} \text{ for all } x, z \in L.$$

Since R, is a semiprime ring, the last expression give that

$$xd(z) = 0 \text{ for all } x, z \in L.$$

That is,

$$Ld(L) = \{0\}.$$

Thus, for any $x, y \in L$,

$$F(xy) = F(x)y + xd(y) = F(x)y, \text{ since } Ld(L) = 0.$$

Then (5.1) implies that

$$\begin{aligned} 0 &= F(xy) - xy \\ &= F(x)y - xy \\ &= (F(x) - x)y \text{ for all } x, y \in L. \end{aligned}$$

This implies that

$$y(F(x) - x)Ry(F(x) - x) = \{0\} \text{ for all } x, y \in L.$$

Since R is semiprime, we get

$$y(F(x) - x) = 0 \quad \text{for all } x, y \in L.$$

Thus for any $x, y \in L$, from above equation we get

$$(F(x) - x)y = 0 \text{ and } y(F(x) - x) = 0, \text{ since } xy = 0, \Rightarrow yx = 0 \text{ for all } x, y \in L.$$

Together implies

$$\begin{aligned} 0 &= F(x)y - yF(x) - (xy - yx) \\ &= [F(x), y] - [x, y] \text{ for all } x, y \in L. \end{aligned}$$

Replacing y with x in above equation we obtain

$$[F(x), x] - [x, x] = 0 \text{ since } [x, x] = 0 \text{ for all } x \in L.$$

Then, we obtain $[F(x), x] = 0$ for all $x \in L$. Then F is commuting on L.

By the second hypothesis, we have

$$F(xy) + xy = 0 \text{ for all } x, y \in L. \tag{5.4}$$

Replacing y with $yz, z \in L$ in (5.4) we get

$$F(xyz) + xyz = 0 \text{ for all } x, y, z \in L.$$

Since $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, it follows that,

$$0 = F(xy)z + xyd(z) + xyz = (F(xy) + xy)z + xyd(z) \text{ for all } x, y, z \in L. \tag{5.5}$$

Application of (5.4) yields that

$$xyd(z) = 0 \text{ for all } x, y, z \in L \tag{5.6}$$

Replacing y with $d(z)ry$, in (5.6) where $r \in R$, we get

$$xd(z)ryd(z) = 0.$$

In particular $y = x$, it follows that

$$xd(z)Rxd(z) = \{0\} \text{ for all } x, z \in L.$$

Since R, is a semiprime ring, the last expression gives that

$$xd(z) = 0 \text{ for all } x, z \in L.$$

That is,

$$Ld(L) = \{0\}.$$

Thus, for any $x, y \in L$,

$$F(xy) = F(x)y + xd(y) = F(x)y, \text{ since } Ld(L) = 0.$$

Then (5.4) implies that

$$0 = F(xy) + xy$$
$$= F(x)y + xy$$
$$= (F(x) + x)y \text{ for all } x, y \in L.$$

This implies that

$$y(F(x) + x)Ry(F(x) + x) = \{0\} \text{ for all } x, y \in L.$$

Since R is semiprime, we get

$$y(F(x) + x) = 0 \text{ for all } x, y \in L.$$

Thus for any $x, y \in L$, from above equation we get

$$(F(x) + x)y = 0 \text{ and } y(F(x) + x) = 0, \text{ since } xy = 0, \Rightarrow yx = 0 \text{ for all } x, y \in L.$$

Together implies

$$0 = F(x)y - yF(x) + xy - yx$$
$$= [F(x), y] + [x, y] \text{ for all } x, y \in L.$$

Replacing y with x in above equation we obtain

$$[F(x), x] + [x, x] = 0 \text{ since } [x, x] = 0 \text{ for all } x \in L.$$

Then we obtain $[F(x), x] = 0$ for all $x \in L$. Then F is commuting on L.

Theorem 5.2 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R \to R$ be a multiplicative (generalized)-derivation associated with the map $d: R \to R$. If $F(xy) \pm yx = 0$ for all $x, y \in L$, then $L[L, L] = \{0\}$, $Ld(L) = \{0\}$, $F(xy) = F(x)y$ for all $x, y \in L$ and F is commuting map on left ideal L.

Proof. By first hypothesis, we have

$$F(xy) - yx = 0 \text{ for all } x, y \in L. \tag{5.7}$$

Replacing y with $yz, z \in L$ in (5.7) we get

$$F(xyz) - yzx = 0 \text{ for all } x, y, z \in L.$$

Since F is multiplicative (generalized)-derivation, we get

$$F(xy)z + xyd(z) - yzx = (F(xy) - yx)z + y[x, z] + xyd(z) = 0 \text{ for all } x, y, z \in L. \tag{5.8}$$

By (5.7), it reduces to

$$y[x, z] + xyd(z) = 0 \text{ for all } x, y, z \in L. \tag{5.9}$$

Now replacing z with x in (5.9), we get

$$y[x, x] + xyd(x) = xyd(x) = 0, \text{since } y[x, x] = 0 \text{ for all } x, y \in L.$$

Since L is a left ideal of R, it follows that

$$xRyd(x) = \{0\} \text{ for all } x, y \in L.$$

In particular $y = x$, which implies

$$xd(x)Rxd(x) = \{0\} \text{ for all } x \in L.$$

Since R is semiprime, the last relation yields that

$$xd(x) = 0, \text{ for all } x \in L.$$

Substituting z for y in (5.9) and then using the fact that $xd(x) = 0$ for all $x \in L$, we obtain

$$y[x, y] + xyd(y) = y[x, y] = 0 \text{ for all } x, y \in L.$$

Replace x by zx in the above equation we get

$$\begin{aligned} 0 &= y[zx, y] \\ &= yz[x, y] + y[z, y]x \\ &= yz[x, y] \quad \text{for all } x, y, z \in L. \end{aligned}$$

This implies

$$[x, y]z[x, y] = 0 \text{ for all } x, y, z \in L.$$

It follows that

$$z[x,y]Rz[x,y] = \{0\} \text{ for all } x,y,z \in L.$$

The semiprimeness of R gives that

$$z[x,y] = 0 \text{ for all } x,y,z \in L.$$

That is,

$$L[L,L] = \{0\}.$$

Then (5.9) gives that

$$xyd(z) = 0 \text{ for all } x,y,z \in L.$$

Which implies,

$$xRyd(z) = \{0\} \text{ for all } x,y,z \in L.$$

And hence

$$xd(z)Rxd(z) = \{0\} \text{ for all } x,y,z \in L.$$

Since R is semiprime, the above expression yields that

$$xd(z) = 0 \text{ for all } x,y \in L.$$

That is,

$$Ld(L) = \{0\}.$$

Then for any $x,y \in L$, we have

$$F(xy) = F(x)y + xd(y) = F(x)y, \text{ since } Ld(L) = 0.$$

Therefore, (5.7) becomes

$$F(x)y - yx = 0 \text{ for all } x,y \in L. \tag{5.10}$$

Now replacing y with xy in (5.10), we obtain

$$F(x)xy - xyx = 0 \text{ for all } x,y \in L. \tag{5.11}$$

Left multiplying (5.10) by x and then subtracting it from (5.11), we get

$$\begin{aligned} 0 &= F(x)xy - xyx - (xF(x)y - xyx) \\ &= F(x)xy - xF(x)y \\ &= \big(F(x)x - xF(x)\big)y \\ &= [F(x),x]y \quad \text{for all } x,y \in L. \end{aligned}$$

Since L is a left ideal of R, we have

$$[F(x),x]Ry = \{0\} \text{ for all } x,y \in L. \tag{5.12}$$

Replacing y with x and then multiplying (5.12) from the right by $F(x)$ we find that

$$[F(x),x]RxF(x) = \{0\} \text{ for all } x \in L. \tag{5.13}$$

Replacing y by $F(x)x$ in (5.12) we get

$$[F(x), x]RF(x)x = \{0\} \text{ for all } x \in L. \tag{5.14}$$

Subtracting (5.13) from (5.14), we find that

$$\begin{aligned}\{0\} &= [F(x), x]RF(x)x - \big([F(x)x, x]RxF(x)\big)\\ &= [F(x), x]R(F(x)x - xF(x))\\ &= [F(x), x]R[F(x), x] \text{ for all } x \in L. \end{aligned} \tag{5.15}$$

The semiprimeness of R gives that $[F(x), x] = 0$ for all $x \in L$. Then F is commuting on L.

By second hypothesis, we have

$$F(xy) + yx = 0 \text{ for all } x, y \in L. \tag{5.16}$$

Replacing y with $yz, z \in L$ in (5.16) we get

$$F(xyz) + yzx = 0 \text{ for all } x, y, z \in L.$$

Since F is multiplicative (generalized)-derivation, we get

$$0 = F(xy)z + xyd(z) + yzx = (F(xy) + yx)z - y[x, z] + xyd(z) \text{ for all } x, y, z \in L. \tag{5.17}$$

By (5.16), it reduces to

$$-y[x, z] + xyd(z) = 0 \text{ for all } x, y, z \in L. \tag{5.18}$$

Now replacing z with x in (5.18) we get

$$-y[x, x] + xyd(x) = xyd(x) = 0, \text{since} - y[x, x] = 0 \text{ for all } x, y, z \in L.$$

Since L is a left ideal of R it follows that

$$xRyd(x) = \{0\} \text{ for all } x, y \in L.$$

In particular $y = x$, which implies

$$xd(x)Rxd(x) = \{0\} \text{ for all } x \in L.$$

Since R is semiprime, the last relation yields that

$$xd(x) = 0 \text{ for all } x \in L.$$

Substituting z for y in (5.18) and then using the fact that $xd(x) = 0$ for all $x \in L$, we obtain

$$-y[x, y] + xyd(y) = -y[x, y] = 0 \text{ for all } x, y, z \in L.$$

Replace x by zx to get

$$\begin{aligned}0 &= -y[zx, y]\\ &= -(yz[x, y] + y[z, y]x)\\ &= yz[x, y] \text{ for all } x, y, z \in L.\end{aligned}$$

This implies that

$$[x,y]z[x,y] = 0 \text{ for all } x,y,z \in L.$$

It follows that

$$z[x,y]Rz[x,y] = \{0\} \text{ for all } x,y,z \in L.$$

The semiprimeness of R gives that

$$z[x,y] = 0 \text{ for all } x,y,z \in L.$$

That is,

$$L[L,L] = \{0\}.$$

Then (5.18) gives that

$$xyd(z) = 0 \text{ for all } x,y,z \in L.$$

Which implies

$$xRyd(z) = \{0\} \text{ for all } x,y,z \in L.$$

And hence

$$xd(z)Rxd(z) = \{0\} \text{ for all } x,z \in L.$$

Since R is semiprime, the above expression yields that

$$xd(z) = 0 \text{ for all } x,z \in L.$$

That is,

$$Ld(L) = \{0\}.$$

Then for any $x,y \in L$, we have

$$F(xy) = F(x)y + xd(y) = F(x)y, \text{ since } Ld(L) = 0.$$

Therefore, (5.16) becomes

$$F(x)y + yx = 0 \text{ for all } x,y \in L. \tag{5.19}$$

Now replacing y with xy in (5.19), we obtain

$$F(x)xy + xyx = 0 \text{ for all } x,y \in L. \tag{5.20}$$

Left multiplying (5.19) by x and then subtracting it from (5.20), we get

$$\begin{aligned} 0 &= F(x)xy + xyx - (xF(x)y + xyx) \\ &= F(x)xy - xF(x)y \\ &= \big(F(x)x - xF(x)\big)y \\ &= [F(x),x]y \text{ for all } x,y \in L. \end{aligned}$$

Since L is a left ideal of R, we have

$$[F(x), x]Ry = \{0\} \text{ for all } x, y \in L. \quad (5.21)$$

Replacing y with x and then multiplying (5.21) from the right by $F(x)$ we find that

$$[F(x), x]RxF(x) = \{0\} \text{ for all } x \in L. \quad (5.22)$$

Replacing y by $F(x)x$ in (5.21) we get

$$[F(x), x]RF(x)x = \{0\} \text{ for all } x, y \in L. \quad (5.23)$$

Subtracting (5.22) from (5.23), we find that

$$\begin{aligned} \{0\} &= [F(x), x]RF(x)x - [F(x), x]RxF(x) \\ &= [F(x), x]R(F(x)x - xF(x)) \\ &= [F(x), x]R[F(x), x] \text{ for all } x \in L. \end{aligned} \quad (5.24)$$

The semiprimeness of R gives that $[F(x), x] = 0$ for all $x \in L$. Then F is commuting on L.

Theorem 5.3 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R \to R$ be a multiplicative (generalized)-derivation associated with the map $d: R \to R$. If $F(x)F(y) \pm xy = 0$ for all $x, y \in L$, then $Ld(L) = \{0\}, F(xy) = F(x)y$ for all $x, y \in L$ and $L[F(x), x] = \{0\}$ for all $x \in L$.

Proof. By first hypothesis, we have

$$F(x)F(y) - xy = 0 \text{ for all } x, y \in L. \quad (5.25)$$

Substituting yz for y in (5.25) and then using the given hypothesis, we find that

$$\begin{aligned} 0 &= F(x)F(yz) - xyz \\ &= F(x)(F(y)z + yd(z)) - xyz \\ &= (F(x)F(y) - xy)z + F(x)yd(z) \text{ for all } x, y, z \in L. \end{aligned} \quad (5.26)$$

Using (5.25), it reduces to

$$F(x)yd(z) = 0 \text{ for all } x, y, z \in L.$$

Replace x by $ux, u \in L$, in the above equation we get

$$\begin{aligned} 0 &= F(ux)yd(z) \\ &= \big(F(u)x + ud(x)\big)yd(z) \\ &= F(u)xyd(z) + ud(x)yd(z) \text{ for all } x, y, z \in L. \end{aligned}$$

Using the fact that $F(x)yd(z) = 0$ for all $x, y, z \in L$, it gives

$$ud(x)yd(z) = 0 \text{ for all } x, y, z, u \in L.$$

Since L is a left ideal, it follows that

$$Ld(L)RLd(L) = \{0\}.$$

Since R is semiprime, we conclude that

$$Ld(L) = \{0\}.$$

Thus for any $x, y \in L$, we obtain

$$F(xy) = F(x)y + xd(y) = F(x)y, \text{ since } Ld(L) = 0.$$

Replacing x by xy in (5.25), we get

$$F(xy)F(y) - xy^2 = 0 \quad (5.27)$$

that is,

$$F(x)yF(y) - xy^2 = 0 \text{ for all } x, y \in L. \quad (5.28)$$

Right multiplying (5.25) by y and then subtracting it from (5.28), we get

$$\begin{aligned} 0 &= F(x)yF(y) - xy^2 - (F(x)F(y)y - xy^2) \\ &= F(x)yF(y) - F(x)F(y)y \\ &= F(x)(yF(y) - F(y)y) \\ &= F(x)[F(y), y] \text{ for all } x, y \in L. \end{aligned}$$

Replacing xz for x in the last relation, we obtain

$$F(x)z[F(y), y] = 0, \text{ since } F(xz) = F(x)z \text{ for all} x, y, z \in L.$$

This implies that

$$[F(x), x]z[F(x), x] = 0.$$

That is,

$$L[F(x), x]RL[F(x), x] = \{0\}.$$

Since R is semiprime, it follows that $L[F(x), x] = \{0\}$ for all $x \in L$, then we obtain $[F(x), x] = 0$, since L is nonzero left ideal of R, then F is commuting map on R.

By Second hypothesis, we have

$$F(x)F(y) + xy = 0 \text{ for all } x, y \in L. \quad (5.29)$$

Substituting yz for y in (5.29) and then using the given hypothesis, we find that

$$\begin{aligned} 0 &= F(x)F(yz) + xyz \\ &= F(x)(F(y)z + yd(z)) + xyz \end{aligned}$$

$$= (F(x)F(y) + xy)z + F(x)yd(z,) \text{ for all } x, y, z \in L. \quad (5.30)$$

Using (5.29), it reduces to

$$F(x)yd(z) = 0 \text{ for all } x, y, z \in L.$$

Replace x by $ux, u \in L$, to get

$$0 = F(ux)yd(z)$$
$$= \big(F(u)x + ud(x)\big)yd(z)$$
$$= F(u)xyd(z) + ud(x)yd(z) \text{ for all } x, y, z, u \in L.$$

Using the fact that $F(x)yd(z) = 0$ for all $x, y, z \in L$, it gives

$$ud(x)yd(z) = 0 \text{ for all } x, y, u, z \in L.$$

Since L is a left ideal, it follows that

$$Ld(L)RLd(L) = \{0\}.$$

Since R is semiprime, we conclude that

$$Ld(L) = \{0\}.$$

Thus for any $x, y \in L$, we obtain

$$F(xy) = F(x)y + xd(y) = F(x)y, \text{ since } Ld(L) = 0$$

Replacing x by xy in (5.29), we get

$$F(xy)F(y) + xy^2 = 0 \quad (5.31)$$

that is,

$$F(x)yF(y) + xy^2 = 0 \text{ for all } x, y \in L. \quad (5.32)$$

Right multiplying (5.29) by y and then subtracting it from (5.32), we get

$$0 = F(x)yF(y) + xy^2 - (F(x)F(y)y + xy^2)$$
$$= F(x)yF(y) - F(x)F(y)y$$
$$= F(x)(yF(y) - F(y)y)$$
$$= F(x)[F(y), y] \text{ for all } x, y \in L.$$

Replacing xz for x in the last relation, we obtain

$$F(x)z[F(y), y] = 0 \text{ since } F(xz) = F(x)z \text{ for all } x, y, z \in L.$$

This implies that

$$[F(x), x]z[F(x), x] = 0 \text{ for all } x, z \in L.$$

That is,

$$L[F(x),x]RL[F(x),x] = \{0\}.$$

Since R is semiprime, it follows that $L[F(x),x] = \{0\}$ for all $x \in L$, then we obtain $[F(x),x] = 0$, since L is nonzero left ideal of R then F is commuting map on R.

The following theorems (Theorem 5.4, Theorem 5.5, Theorem 5.6, and Theorem 5.7) gives the concepts essentiality of semiprime ring with multiplicative (generalized)-derivation.

Theorem 5.4 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R \to R$ be a multiplicative (generalized)-derivation associated with the map $d: R \to R$. If $F(xy) \pm xy \in Z(R)$ for all $x, y \in L$, then $L[d(x),x] = \{0\}$ for all $x \in L$.

Proof. By first hypothesis, we have

$$F(xy) - xy \in Z(R) \text{ for all } x, y \in L. \tag{5.33}$$

Now we replace y with yz in (5.33), where $z \in L$ and then we get

$$\begin{aligned} F(xyz) - xyz &= F(xy)z + xyd(z) - xyz \\ &= (F(xy) - xy)z + xyd(z) \in Z(R) \text{ for all } x, y, z \in L. \end{aligned} \tag{5.34}$$

Applying (5.33) to (5.34) and commuting both sides by z yields

$$[xyd(z),z] = 0 \text{ for all } x, y, z \in L. \tag{5.35}$$

Substituting rx for x in (5.35), where $r \in R$, and using (5.35) we obtain

$$\begin{aligned} 0 &= [rxyd(z),z] \\ &= r[xyd(z),z] + [r,z]xyd(z) \\ &= [r,z]xyd(z) \end{aligned} \tag{5.36}$$

Replacing x with $d(z)x$ in (5.36) we get

$$[r,z]d(z)xyd(z) = 0 \tag{5.37}$$

Which implies

$$[r,z]d(z)Rxyd(z) = \{0\} \text{ for all } x, y, z \in L \text{ and } r \in R.$$

Interchanging x and y in the above expressions, and then subtracting one from the other, we get,

$$\{0\} = [r,z]d(z)Ryxd(z) - [r,z]d(z)Rxyd(z)$$

$$= [r, z]d(z)R(yx - xy)d(z)$$

$$= [r, z]d(z)R[x, y]d(z) \text{ for all } x, y, z \in L.$$

In particular $y = z$, $r = x$ in the above equation we get

$$[x, z]d(z)R[x, z]d(z) = \{0\} \text{ for all } x, z \in L.$$

The semiprimeness of R yields that

$$[x, z]d(z) = 0 \text{ for all } x, z \in L. \tag{5.38}$$

Right multiplying (5.38) by z we get

$$[x, z]d(z)z = 0 \text{ for all } x, z \in L. \tag{5.39}$$

Replace x by xz in (5.38) we get

$$\begin{aligned} 0 &= [xz, z]d(z) \\ &= x[z, z]d(z) + [x, z]zd(z) \\ &= [x, z]zd(z) \text{ since } [z, z] = 0 \text{ for all } x, z \in L. \end{aligned} \tag{5.40}$$

Now (5.39) and (5.40) together implies that

$$\begin{aligned} 0 &= [x, z]d(z)z - [x, z]zd(z) \\ &= [x, z](d(z)z - zd(z)) \\ &= [x, z][d(z), z] \text{ for all } x, z \in L. \end{aligned} \tag{5.41}$$

Replacing x by $d(z)x$ in the last expression, we obtain

$$[d(z)x, z][d(z), z] = [d(z), z]x[d(z), z] + d(z)[x, z][d(z), z] = 0 \text{ for all } x, y, z \in L.$$

Application of (5.41) yields that

$$[d(z), z]x[d(z), z] = 0 \text{ for all } x, z \in L.$$

That is

$$L[d(z), z]RL[d(z), z] = \{0\} \text{ for all } x, z \in L.$$

Hence, the semiprimeness of R gives that $L[d(z), z] = \{0\}$ for all $z \in L$.

By second hypothesis, we have

$$F(xy) + xy \in Z(R) \text{ for all } x, y \in L. \tag{5.42}$$

Now we replace y with yz in (5.42), where $z \in L$ and then we get

$$F(xyz) + xyz = F(xy)z + xyd(z) + xyz$$
$$= (F(xy) + xy)z + xyd(z) \in Z(R) \text{ for all } x, y, z \in L. \tag{5.43}$$

Applying (5.42) to (5.43) and commuting both sides by z yields

$$[xyd(z), z] = 0 \text{ for all } x, y, z \in L. \tag{5.44}$$

Substituting rx for x in (5.44), where $r \in R$, and using (5.44) we obtain

$$0 = [rxyd(z), z]$$
$$= r[xyd(z), z] + [r, z]xyd(z)$$
$$= [r, z]xyd(z). \tag{5.45}$$

Replacing x with $d(z)x$ in (5.45) we get

$$[r, z]d(z)xyd(z) = 0 \tag{5.46}$$

Which implies

$$[r, z]d(z)Rxyd(z) = \{0\} \text{ for all } x, y, z \in L \text{ and } r \in R.$$

Interchanging x and y in the last expression, and then subtracting one from the other, we get

$$\{0\} = [r, z]d(z)Ryxd(z) - [r, z]d(z)Rxyd(z)$$
$$= [r, z]d(z)R(yx - xy)d(z)$$
$$= [r, z]d(z)R[x, y]d(z) \text{ for all } x, y, z \in L.$$

In particular, $y = z, r = x$, in the above equation we get,

$$\{0\} = [x, z]d(z)R[x, z]d(z) \text{ for all } x, z \in L.$$

The semiprimeness of R yields that

.
$$[x, z]d(z) = 0 \text{ for all } x, z \in L. \tag{5.47}$$

Right multiplying (5.47) by z we get

$$[x, z]d(z)z = 0 \text{ for all } x, z \in L. \tag{5.48}$$

Replace x by xz in (5.47) to get

$$0 = [xz, z]d(z)$$
$$= x[z, z]d(z) + [x, z]zd(z)$$
$$= [x, z]zd(z) \text{ for all } x, z \in L. \tag{5.49}$$

Now (5.48) and (5.49) together implies that

$$0 = [x, z]d(z)z - [x, z]zd(z)$$
$$= [x, z](d(z)z - zd(z))$$
$$= [x, z][d(z), z] = 0 \text{ for all } x, z \in L. \tag{5.50}$$

Replacing x by $d(z)x$ in the last expression, we obtain

$$[d(z)x,z][d(z),z]=[d(z),z]x[d(z),z]+d(z)[x,z][d(z),z]=0 \text{ for all } x,z\in L.$$

Application of (5.50) yields that

$$[d(z),z]x[d(z),z]=0 \text{ for all } x,z\in L.$$

That is

$$L[d(z),z]RL[d(z),z]=\{0\} \text{ for all } z\in L.$$

Hence, the semiprimeness of R gives that $L[d(z),z]=\{0\}$ for all $z\in L$.

Theorem 5.5 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R\to R$ be a multiplicative (generalized)-derivation associated with the map $d: R\to R$. If $F(xy)\pm yx\in Z(R)$ for all $x,y\in L$, then $x[x,L]\subseteq Z(R)$ for all $x\in L$ and $L[d(x),x]=\{0\}$ for all $x\in L$.

Proof. By first hypothesis, we have

$$F(xy)-yx\in Z(R) \text{ for all } x,y\in L. \tag{5.51}$$

In the above relation, replacing x with xy and y with y^2, respectively and then subtracting one from the other, we obtain

$$(F(xy^2)-yxy)-(F(xy^2)-y^2x)\in Z(R) \tag{5.52}$$

This implies that

$$\begin{aligned}-yxy+y^2x&=y(-xy+yx)\\&=y[y,x]\in Z(R) \text{ for all } x,y\in L.\end{aligned}$$

Thus for all $x\in L$, $x[x,L]\subseteq Z(R)$. Now substituting yz for y in (5.51), where $z\in L$, we get

$$\begin{aligned}F(xyz)-(yz)x&=F(xy)z+xyd(z)-yzx\\&=(F(xy)-yx)z+y[x,z]+xyd(z)\in Z(R)\end{aligned} \tag{5.53}$$

Commuting both sides of (5.53) with z and then using (5.51), we obtain

$$[y[x,z],z]+[xyd(z),z]=0 \text{ for all } x,y,z\in L. \tag{5.54}$$

Replacing x with xz in (5.54), we get

$$\begin{aligned}0&=[y[xz,z],z]+[xzyd(z),z]\\&=[y(x[z,z]+[x,z]z),z]+[xzyd(z),z]\end{aligned}$$

$$= [yx[z,z],z] + [y[x,z]z,z] + [xzyd(z),z]$$
$$= [y[x,z]z,z] + [xzyd(z),z]$$
$$= y[x,z][z,z] + [y[x,z],z]z + [xzyd(z),z]$$
$$= [y[x,z],z]z + [xzyd(z),z] \text{ since } [z,z] = 0 \text{ forall } x,y \in L. \qquad (5.55)$$

Right multiplying (5.54) by z and then subtracting it from (5.55), we get

$$0 = [y[x,z],z]z + [xzyd(z),z] - ([y[x,z],z]z + [xyd(z),z]z)$$
$$= [xzyd(z),z] - [xyd(z),z]z$$
$$= [x([zyd(z),z] - [yd(z),z]z)]$$
$$= [x(z[yd(z),z] + [z,z]yd(z) - [yd(z),z]z)]$$
$$= [x(z[yd(z),z] - [yd(z),z]z)]$$
$$= [x[yd(z),z],z] \text{ for all } x,y,z \in L. \qquad (5.56)$$

Replacing x with $rx, r \in R$ in the above relation and then using (5.56), we have

$$0 = [rx[yd(z),z],z]$$
$$= r[x[yd(z),z],z] + [r,z]x[yd(z),z]$$
$$= [r,z]x[yd(z),z]. \qquad (5.57)$$

In particular, for $r = yd(z)$, in the above equation, we get

$$[yd(z),z]x[yd(z),z] = 0 \text{ for all } x,y,z \in L.$$

Since L is a left ideal of R, it follows that

$$x[yd(z),z]Rx[yd(z),z] = \{0\} \text{ for all } x,y,z \in L.$$

Since R is semiprime, we get

$$x[yd(z),z] = 0 \text{ for all } x,y,z \in L.$$

Now replacing y with $d(z)y$, we get

$$x[d(z)yd(z),z] = 0 \qquad (5.58)$$

that is,

$$x\big(d(z)yd(z)z - zd(z)yd(z)\big) = 0 \text{ for all } x,y,z \in L. \qquad (5.59)$$

Now we replacing y with $yd(z)u$, where $u \in L$, and then obtain

$$x\big(d(z)yd(z)ud(z)z - zd(z)yd(z)ud(z)\big) = 0 \text{ for all } x,y,z \in L. \qquad (5.60)$$

By (5.59), this can be written as

$$x\big(d(z)yzd(z)ud(z) - d(z)yd(z)zud(z)\big) = xd(z)y(zd(z) - d(z)z)ud(z) = 0 \qquad (5.61)$$

that is,

$$xd(z)y[d(z),z]ud(z) = 0 \text{ for all } x,y \in L. \qquad (5.62)$$

This implies that,

$$x[d(z),z]y[d(z),z]u[d(z),z] = 0 \text{ for all } x,y,z \in L.$$

and so $$(L[d(z),z])^3 = \{0\} \text{ for all } z \in L.$$

Since a semiprime ring contains no nonzero nilpotent left ideals it follows that

$$L[d(z),z] = \{0\} \text{ for all } z \in L.$$

By second hypothesis, we have

$$F(xy) + yx \in Z(R) \text{ for all } x,y \in L. \qquad (5.63)$$

In the above relation, replacing x with xy and y with y^2, respectively and then subtracting one from the other, we obtain

$$(F(xy^2) + yxy) - (F(xy^2) + y^2x) \in Z(R) \qquad (5.64)$$

This implies that

$$yxy - y^2x = y(xy - yx)$$
$$= y[x,y] \in Z(R) \text{ for all } x,y \in L.$$

Thus for all $x \in L$, $x[x,L] \subseteq Z(R)$. Now substituting yz for y in (5.63), where $z \in L$, we get

$$F(xyz) + (yz)x = F(xy)z + xyd(z) + yzx$$
$$= (F(xy) + yx)z - y[x,z] + xyd(z) \in Z(R) \qquad (5.65)$$

Commuting both sides of (5.65) with z and then using (5.63), we obtain

$$-[y[x,z],z] + [xyd(z),z] = 0 \text{ for all } x,y,z \in L. \qquad (5.66)$$

Replacing x with xz in (5.66), we get

$$0 = -[y[xz,z],z] + [xzyd(z),z]$$
$$= -([y(x[z,z] + [x,z]z),z]) + [xzyd(z),z]$$
$$= -[y[x,z]z,z] + [xzyd(z),z]$$
$$= -y[x,z][z,z] - [y[x,z],z]z + [xzyd(z),z]$$
$$= -[y[x,z],z]z + [xzyd(z),z] \text{ for all } x,y,z \in L. \qquad (5.67)$$

Right multiplying (5.66) by z and then subtracting it from (5.67), we get

$$0 = -[y[x,z],z]z + [xzyd(z),z] - (-[y[x,z],z]z + [xyd(z),z]z)$$
$$= [xzyd(z),z] - [xyd(z),z]z$$
$$= [x([zyd(z),z] - [yd(z),z]z)]$$

$$= [x(z[yd(z), z] + [z, z]yd(z) - [yd(z), z]z)]$$
$$= [x(z[yd(z), z] - [yd(z), z]z)]$$
$$= [x[yd(z), z], z] \quad \text{for all } x, y, z \in L. \tag{5.68}$$

Replacing x with rx, $r \in R$ in the above relation and then using (5.68), we have

$$0 = [rx[yd(z), z], z]$$
$$= r[x[yd(z), z], z] + [r, z]x[yd(z), z]$$
$$= [r, z]x[yd(z), z]. \tag{5.69}$$

In particular, for $r = yd(z)$, in the above equation, we get

$$[yd(z), z]x[yd(z), z] = 0 \text{ for all } x, y, z \in L.$$

Since L is a left ideal of R, it follows that

$$x[yd(z), z]Rx[yd(z), z] = \{0\} \text{ for all } x, y, z \in L.$$

Since R is semiprime, we get

$$x[yd(z), z] = 0 \text{ for all } x, y, z \in L.$$

Now replacing y with $d(z)y$, we get

$$x[d(z)yd(z), z] = 0 \tag{5.70}$$

that is,

$$x(d(z)yd(z)z - zd(z)yd(z)) = 0 \text{ for all } x, y, z \in L. \tag{5.71}$$

Now we put $y = yd(z)u$, where $u \in L$, and then obtain

$$x(d(z)yd(z)ud(z)z - zd(z)yd(z)ud(z)) = 0 \text{ for all } x, y, z \in L. \tag{5.72}$$

By (5.71), this can be written as

$$x(d(z)yzd(z)ud(z) - d(z)yd(z)zud(z)) = xd(z)y(zd(z) - d(z)z)ud(z) = 0 \tag{5.73}$$

that is,

$$xd(z)y[d(z), z]ud(z) = 0 \text{ for all } x, y, z \in L \tag{5.74}$$

This implies $x[d(z), z]y[d(z), z]u[d(z), z] = 0$ for all $x, y, z \in L$ and so $(L[d(z), z])^3 = \{0\}$ for all $z \in L$. Since a semiprime ring contains no nonzero nilpotent left ideals it follows that $L[d(z), z] = \{0\}$ for all $z \in L$.

Theorem 5.6 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R \to R$ be a multiplicative (generalized)-derivation associated with the map $d: R \to R$. If $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in L$, then $L[d(x), x] = \{0\}$ for all $x \in L$.

Proof. By first hypothesis, we have

$$F(x)F(y) - xy \in Z(R) \text{ for all } x, y \in L. \quad (5.75)$$

Replacing y with $yz, z \in L$, we have

$$F(x)F(yz) - x(yz) \in Z(R) \quad (5.76)$$

which gives

$$F(x)(F(y)z + yd(z)) - xyz = F(x)(F(y) - xy)z + F(x)yd(z) \in Z(R), \forall x, y \in L. \quad (5.77)$$

Commuting both sides of (5.77) with z and then using (5.75), we get

$$[F(x)yd(z), z] = 0 \text{ for all } x, y, z \in L. \quad (5.78)$$

Replacing y with zy in the above relation we obtain

$$[F(x)zyd(z), z] = 0 \text{ for all } x, y \in L. \quad (5.79)$$

Now replacing x with xz in (5.78), we get

$$\begin{aligned} 0 &= [F(xz)yd(z), z] \\ &= \left[\left(F(x)z + xd(z)\right)yd(z), z\right] \\ &= [F(x)zyd(z), z] + [xd(z)yd(z), z] \text{ for all } x, y, z \in L. \quad (5.80) \end{aligned}$$

Using (5.79), the above relation reduces to

$$[xd(z)yd(z), z] = 0 \text{ for all} x, y, z \in L. \quad (5.81)$$

In (5.81), we replace x with $d(z)x$ and then using (5.81) we obtain

$$\begin{aligned} 0 &= [d(z)xd(z)yd(z), z] \\ &= d(z)[xd(z)yd(z), z] + [d(z), z]xd(z)yd(z) \\ &= [d(z), z]xd(z)yd(z) \text{ for all } x, y, z \in L. \quad (5.82) \end{aligned}$$

This implies

$$[d(z), z]x[d(z), z]y[d(z), z] = 0 \text{ for all } x, y \in L.$$

That is,

$$(L[d(z), z])^3 = \{0\} \text{ for all } z \in L.$$

Since R is semiprime, it contains no nonzero nilpotent left ideals, implying

$$L[d(z), z] = \{0\} \text{ for all } z \in L.$$

By second hypothesis, we have

$$F(x)F(y) + xy \in Z(R) \text{ for all } x, y \in L. \tag{5.83}$$

Replacing y with $yz, z \in L$, we have

$$F(x)F(yz) + x(yz) \in Z(R) \tag{5.84}$$

which gives

$$F(x)\big(F(y)z + yd(z)\big) + xyz = F(x)(F(y) + xy)z + F(x)yd(z) \in Z(R), \forall x, y \in L \tag{5.85}$$

Commuting both sides of (5.85) with z and then using (5.83), we get

$$[F(x)yd(z), z] = 0 \text{ for all } x, y, z \in L. \tag{5.86}$$

Replacing y with zy in the above relation we obtain

$$[F(x)zyd(z), z] = 0 \text{ for all } x, y, z \in L. \tag{5.87}$$

Now replacing x with xz in (5.86), we get

$$\begin{aligned} 0 &= [F(xz)yd(z), z] \\ &= \big[\big(F(x)z + xd(z)\big)yd(z), z\big] \\ &= [F(x)zyd(z), z] + [xd(z)yd(z), z] \text{ for all } x, y, z \in L. \end{aligned} \tag{5.88}$$

Using (5.87), the above relation reduces to

$$[xd(z)yd(z), z] = 0 \text{ for all } x, y, z \in L. \tag{5.89}$$

In (5.89), we replace x with $d(z)x$ and then using (5.89) we obtain

$$\begin{aligned} 0 &= [d(z)xd(z)yd(z), z] \\ &= d(z)[xd(z)yd(z), z] + [d(z), z]xd(z)yd(z) \\ &= [d(z), z]xd(z)yd(z) \text{ for all } x, y, z \in L. \end{aligned} \tag{5.90}$$

This implies

$$[d(z), z]x[d(z), z]y[d(z), z] = 0 \text{ for all } x, y, z \in L.$$

That is,

$$(L[d(z), z])^3 = \{0\} \text{ for all } z \in L.$$

Since R is semiprime, it contains no nonzero nilpotent left ideals, implying

$$L[d(z), z] = \{0\} \text{ for all } z \in L.$$

Theorem 5.7 (Dhara and Ali, 2013) Let R be a semiprime ring, L be a nonzero left ideal of R and $F: R \to R$ be a multiplicative (generalized)-derivation associated with the map $d: R \to R$. If $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in L$, then $L[d(x), x] = \{0\}$, for all $x \in L$.

Proof. By the first hypothesis, we have

$$F(x)F(y) - yx \in Z(R) \text{ for all } x, y \in L. \quad (5.91)$$

Replacing y with yz we get

$$F(x)F(yz) - yzx \in Z(R) \text{ for all } x, y, z \in L. \quad (5.92)$$

This give

$$F(x)F(yz) - yzx = F(x)\big(F(y)z + yd(z)\big) - yzx$$

$$= (F(x)F(y) - yx)z + y[x, z] + F(x)yd(z) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.93)$$

Commuting both sides of (5.93) with z and then using (5.91), it reduces to

$$[y[x, z], z] + [F(x)yd(z), z] = 0 \quad \text{for all } x, y \in L. \quad (5.94)$$

Replacing x with xz in the above relation we get

$$0 = [y[xz, z], z] + [F(xz)yd(z), z]$$

$$= [y(x[z, z] + [x, z]z), z] + \big[\big(F(x)z + xd(z)\big)yd(z), z\big]$$

$$= [y[x, z]z, z] + \big[\big(F(x)z + xd(z)\big)yd(z), z\big]$$

$$= y[x, z][z, z] + [y[x, z], z]z + \big[F(x)z + xd(z)\big)yd(z), z\big]$$

$$= [y[x, z], z]z + [F(x)zyd(z), z] + [xd(z)yd(z), z] \text{ for all } x, y, z \in L. \quad (5.95)$$

Replacing y with zy in (5.94), we get

$$0 = [zy[x, z], z] + [F(x)zyd(z), z]$$

$$= z[y[x, z], z] + [z, z]y[x, z] + [F(x)zyd(z), z]$$

$$= z[y[x,z],z] + [F(x)zyd(z),z] \text{ for all } x,y,z \in L. \tag{5.96}$$

Subtracting (5.96) from (5.95), we have

$$0 = [y[x,z],z]z + [F(x)zyd(z),z] + [xd(z)yd(z),z] - (z[y[x,z],z] + [F(x)zyd(z),z])$$

$$= [y[x,z],z]z - (z[y[x,z],z]) + [xd(z)yd(z),z]$$

$$= \big[[y[x,z],z],z\big] + [xd(z)yd(z),z] \text{ for all } x,y,z \in L. \tag{5.97}$$

Replacing x with xz the above relation yields

$$0 = \big[[y[xz,z],z],z\big] + [xzd(z)yd(z),z]$$

$$= [[y(x[z,z] + [x,z]z),z],z] + [xzd(z)yd(z),z]$$

$$= \big[[y[x,z]z,z],z\big] + [xzd(z)yd(z),z]$$

$$= [(y[x,z][z,z] + [y[x,z],z]z),z] + [xzd(z)yd(z),z]$$

$$= [[y[x,z],z]z,z] + [xzd(z)yd(z),z]$$

$$= [y[x,z],z][z,z] + \big[[y[x,z],z],z\big]z + [xzd(z)yd(z),z]$$

$$= \big[[y[x,z],z],z\big]z + [xzd(z)yd(z),z] = 0 \text{ for all } x,y,z \in L. \tag{5.98}$$

Right multiplying (5.97) by z and then subtracting it from (5.98), we get

$$0 = \big[[y[x,z],z],z\big]z + [xzd(z)yd(z),z] - (\big[[y[x,z],z],z\big]z + [xd(z)yd(z),z]z)$$

$$= [xzd(z)yd(z),z] - [xd(z)yd(z),z]z$$

$$= [x(z[d(z)yd(z),z] + [z,z]d(z)yd(z)) - [d(z)yd(z),z]z)]$$

$$= [x(z[d(z)yd(z),z] - [d(z)yd(z),z]z)]$$

$$= [x[d(z)yd(z),z],z] \text{ for all } x,y,z \in L. \tag{5.99}$$

Now we substitute $d(z)yd(z)x$ for x in (5.99) and we get

$$0 = [d(z)yd(z)x[d(z)yd(z), z], z]$$

$$= d(z)yd(z)[x[d(z)yd(z), z], z] + [d(z)yd(z), z]x[d(z)yd(z), z] \quad (5.100)$$

Using (5.99), it reduces to

$$[d(z)yd(z), z]x[d(z)yd(z), z] = 0 \text{ for all } x, y, z \in L. \quad (5.101)$$

Since L is a left ideal, it follows that

$$x[d(z)yd(z), z]Rx[d(z)yd(z), z] = \{0\} \text{ for all } x, y, z \in L.$$

And hence

$$x[d(z)yd(z), z] = 0 \text{ for all } x, y, z \in L. \quad (5.102)$$

That is,

$$x(d(z)yd(z)z - zd(z)yd(z) = 0 \text{ for all } x, y, z \in L. \quad (5.103)$$

Now we put $y = yd(z)u$, where $u \in L$, and then obtain

$$x(d(z)yd(z)ud(z)z - zd(z)yd(z)ud(z)) = 0 \text{ for all } x, y, z \in L. \quad (5.104)$$

By (5.103), this can be written as

$$x(d(z)yzd(z)ud(z) - d(z)yd(z)zud(z)) = 0 \quad (5.105)$$

That is,

$$xd(z)y[d(z), z]ud(z) = 0 \text{ for all } x, y, z \in L. \quad (5.106)$$

This implies

$$x[d(z), z]y[d(z), z]u[d(z), z] = 0 \text{ for all } x, y, z \in L.$$

and so

$$(L[d(z), z])^3 = \{0\}, \forall z \in L.$$

Since a semiprime ring contains no nonzero nilpotent left ideals it follows that

$$L[d(z), z] = \{0\} \text{ for all } z \in L.$$

By the second hypothesis, we have

$$F(x)F(y) + yx \in Z(R) \text{ for all } x, y \in L. \quad (5.107)$$

Replacing y with yz, we get

$$F(x)F(yz) + yzx = F(x)\big(F(y)z + yd(z)\big) + yzx \in Z(R) \text{ for all } x, y, z \in L. \quad (5.108)$$

This gives

$$(F(x)F(y) + yx)z - y[x, z] + F(x)yd(z) \in Z(R) \text{ for all } x, y, z \in L. \quad (5.109)$$

Commuting both sides of (5.109) with z and then using (5.107), it reduces to

$$-[y[x, z], z] + [F(x)yd(z), z] = 0 \text{ for all } x, y, z \in L. \quad (5.110)$$

Replacing x with xz in the above relation we get

$$0 = -[y[xz, z], z] + [F(xz)yd(z), z]$$

$$= -[y(x[z, z] + [x, z]z), z] + [\big(F(x)z + xd(z)\big)yd(z), z]$$

$$= -([yx[z, z], z] + [y[x, z]z, z]) + \big[\big(F(x)z + xd(z)\big)yd(z), z\big]$$

$$= -[y[x, z]z, z] + \big[\big(F(x)z + xd(z)\big)yd(z), z\big]$$

$$= -(y[x, z][z, z] + [y[x, z], z]z) + [F(x)zyd(z), z] + [xd(z)yd(z), z]$$

$$= -[y[x, z], z]z + [F(x)zyd(z), z] + [xd(z)yd(z), z] \text{ for all } x, y, z \in L. \quad (5.111)$$

Replacing y with zy in (5.110), we get

$$0 = -[zy[x, z], z] + [F(x)zyd(z), z]$$

$$= -(z[y[x, z], z] + [z, z]y[x, z]) + [F(x)zyd(z), z]$$

$$= -z[y[x, z], z] + [F(x)zyd(z), z] = 0 \text{ for all } x, y, z \in L. \quad (5.112)$$

Subtracting (5.112) from (5.111), we have

$$0 = -[y[x, z], z]z + [F(x)zyd(z), z] + [xd(z)yd(z), z] - (-z[y[x, z], z] + [F(x)zyd(z), z])$$

$$= -[y[x, z], z]z + (z[y[x, z], z]) + [xd(z)yd(z), z]$$

$$= \big[[y[x, z], z], z\big] - [xd(z)yd(z), z] \text{ for all } x, y, z \in L. \quad (5.113)$$

Replacing x with xz the above relation yields

$$0 = \big[[y[xz,z],z],z\big] - [xzd(z)yd(z),z]$$

$$= [[y(x[z,z] + [x,z]z),z],z] - [xzd(z)yd(z),z]$$

$$= \big[[y[x,z]z,z],z\big] - [xzd(z)yd(z),z]$$

$$= [(y[x,z][z,z] - [y[x,z],z]z),z] - [xzd(z)yd(z),z]$$

$$= \big[[y[x,z],z]z,z\big] - [xzd(z)yd(z),z]$$

$$= [y[x,z],z][z,z] - \big[[y[x,z],z],z\big]z - [xzd(z)yd(z),z]$$

$$= \big[[y[x,z],z],z\big]z - [xzd(z)yd(z),z] = 0 \text{ for all } x,y,z \in L. \qquad (5.114)$$

Right multiplying (5.113) by z and then subtracting it from (5.114), we get

$$0 = \big[[y[x,z],z],z\big]z - [xzd(z)yd(z),z] - (\big[[y[x,z],z],z\big]z - [xd(z)yd(z),z]z)$$

$$= [xzd(z)yd(z),z] - [xd(z)yd(z),z]z$$

$$= [x(z[d(z)yd(z),z] + [z,z]d(z)yd(z)) - [d(z)yd(z),z]z)]$$

$$= [x(z[d(z)yd(z),z] - [d(z)yd(z),z]z)]$$

$$= [x[d(z)yd(z),z],z] \text{ for all } x,y,z \in L. \qquad (5.115)$$

Now we substitute $d(z)yd(z)x$ for x in (5.115) and get

$$0 = [d(z)yd(z)x[d(z)yd(z),z],z]$$

$$= d(z)yd(z)[x[d(z)yd(z),z],z] + [d(z)yd(z),z]x[d(z)yd(z),z] \qquad (5.116)$$

Using (5.115), it reduces to

$$0 = [d(z)yd(z),z]x[d(z)yd(z),z] \text{ for all } x,y,z \in L. \qquad (5.117)$$

Since L is a left ideal, it follows that

$$x[d(z)yd(z),z]Rx[d(z)yd(z),z] = \{0\}$$

And hence

$$x[d(z)yd(z),z] = 0 \text{ for all } x,y,z \in L. \tag{5.118}$$

That is,

$$x\big(d(z)yd(z)z - zd(z)yd(z)\big) = 0 \text{ for all } x,y,z \in L \tag{5.119}$$

Now we put $y = yd(z)u$, where $u \in L$, and then obtain

$$x(d(z)yd(z)ud(z)z - zd(z)yd(z)ud(z)) = 0 \text{ for all } x,y,z \in L. \tag{5.120}$$

By (5.119), this can be written as

$$x(d(z)yzd(z)ud(z) - d(z)yd(z)zud(z)) = 0 \tag{5.121}$$

That is,

$$xd(z)y[d(z),z]ud(z) = 0 \text{ for all } x,y,z \in L. \tag{5.122}$$

This implies

$$x[d(z),z]y[d(z),z]u[d(z),z] = 0 \text{ for all } x,y,z \in L.$$

and so

$$(L[d(z),z])^3 = \{0\} \text{ for all } z \in L.$$

Since a semiprime ring contains no nonzero nilpotent left ideals it follows that

$$L[d(z),z] = \{0\} \text{ for all } z \in L.$$

Camci and Aydin (2017) stated and proved some functional identities with multiplicative (generalized)-derivations F and multiplicative left centralizer H under the following conditions. We deals with their results by incorporating further explanatory steps in their proof.

Lemma 5.1 (Camci and Aydin, 2017) Let R be a semiprime ring. If F is a multiplicative (generalized)-derivation associated with the map d, then d is a multiplicative derivation, that is, $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Proof. Since F is a multiplicative (generalized)-derivation we have

$$F\big(x(yz)\big) = F(x)yz + xd(yz) \quad \text{for all } x,y,z \in R.$$

and

$$F\big((xy)z\big) = F(x)yz + xd(y)z + xyd(z) \text{ for all } x, y, z \in R.$$

Hence we get,

$$xd(yz) = xd(y)z + xyd(z) \text{ for all } x, y, z \in R.$$

From the last equation, we find that

$$0 = x\big(d(yz) - d(y)z - yd(z)\big)$$

$$\{0\} = R\big(d(yz) - d(y)z - yd(z)\big) \text{ for all } x, y, z \in R.$$

Since the semiprimeness of R, we have, $d(yz) = d(y)z + yd(z)$ for all $x, y, z \in R$.

Lemma 5.2 (Camci and Aydin, 2017) Let R be a semiprime ring and F be a multiplicative (generalized)-derivation associated with d. If $F(xy) = 0$ holds for all $x, y \in R$, then $F = 0$.

Proof. By the assumption, we have

$$F(xy) = 0 \text{ for all } x, y \in R.$$

If we replace x by xz with $z \in R$, we get

$$F(xzy) = 0 \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we get

$$F(xzy) = F(xz)y + xzd(y) = 0 \text{ for all } x, y, z \in R.$$

Using the hypothesis we find that

$$xzd(y) = 0 \text{ for all } x, y, z \in R.$$

Since R is a semiprime ring, we obtain $xd(y) = 0$, for all $x, y \in R$. This means $d = 0$. From the definition of F, we get $F(xy) = F(x)y$, for all $x, y \in R$. By the hypothesis we see that

$$F(x)y = 0 \text{ for all } x, y \in R.$$

From the semiprimeness of R, we find that $F = 0$.

Lemma 5.3 (Camci and Aydin, 2017) Let R be a semiprime ring and F be a multiplicative (generalized)-derivation associated with d. If $F(xy) \in Z(R)$ holds for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.

Proof. By the hypothesis, we have

$$F(xy) \in Z(R) \text{ for all } x, y \in R.$$

Taking yz instead of y with $z \in R$, we get

$$F(xyz) \in Z(R) \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$F(xy)z + xyd(z) \in Z(R) \text{ for all } x, y, z \in R.$$

Commuting both sides above with z and using the hypothesis, we get

$$[xyd(z), z] = 0 \text{ for all } x, y, z \in R.$$

Replacing x by rx with $r \in R$, so we get

$$\begin{aligned} 0 &= [rxyd(z), z] \\ &= r[xyd(z), z] + [r, z]xyd(z) = 0 \\ &= [r, z]xyd(z) = 0 \text{ since } [xyd(z), z] = 0 \text{ for all } x, y, z, r \in R. \end{aligned}$$

In the above equation replacing x by $d(z)x$, we find that

$$[r, z]d(z)xyd(z) = 0 \text{ for all } x, y, z, r \in R.$$

In particular $r = x, y = z$, this implies that, for all $x, y, s \in R$.

$$[x, y]d(y)s[x, y]d(y) = [x, y]d(y)sxyd(y) - [x, y]d(y)syxd(y) = 0.$$

Since R is a semiprime ring, we find that

$$[x, y]d(y)R[x, y]d(y) = [x, y]d(y) = 0 \text{ for all } x, y \in R.$$

Replacing x by xy with $y \in R$, we have

$$\begin{aligned} 0 &= [xy, y]d(z) \\ &= x[y, y]d(z) + [x, y]yd(z) \\ &= [x, y]yd(y) = 0 \text{ since } [y, y] = 0 \text{ for all } x, y \in R. \end{aligned}$$

Hence, we see that

$$[x, y][d(y), y] = 0 \text{ for all } x, y \in R.$$

If we replace x by $d(y)x$ and using the semiprimeness of R, we get

$$\begin{aligned} 0 &= [d(y)x, y][d(y), y] \\ &= d(y)[x, y][d(y), y] + [d(y), y]x[d(y), y] \\ &= [d(y), y]x[d(y), y] \\ &= [d(y), y] = 0 \text{ for all } y \in R. \end{aligned}$$

Lemma 5.4 (Camci and Aydin, 2017) Let R be a ring, F be a multiplicative (generalized)-derivation associated with d and H be a multiplicative left centralizer. If the map $G: R \to R$ is defined as $G(x) = F(x) \mp H(x)$ for all $x \in R$, then G is multiplicative (generalized)-derivation associated with d.

Proof. We suppose that, for all $x \in R$

$$G(x) = F(x) \mp H(x)$$

So we have, for all $x, y \in R$

$$\begin{aligned} G(xy) &= F(xy) \mp H(xy) \\ &= F(x)y + xd(y) \mp H(x)y \\ &= \big(F(x) \mp H(x)\big)y + xd(y) \\ &= G(x)y + xd(y) \end{aligned}$$

Then G is a multiplicative (generalized)-derivation associated with d.

The following theorems (Theorem 5.8, and Theorem 5.9) gives the concepts on commuting map on semiprime ring R with multiplicative (generalized)-derivation.

Theorem 5.8 (Camci and Aydin, 2017) Let R be a semiprime ring, $F: R \to R$ be a multiplicative (generalized)-derivation associated with d and $H: R \to R$ be a multiplicative left centralizer. If $F(xy) \mp H(yx) = 0$ holds for all $x, y \in R$, then $d = 0$. Moreover, $F(xy) = F(x)y$ holds for all $x, y \in R$ and $[F(x), x] = 0$, for all $x \in R$.

Proof. By the first hypothesis, we have

$$F(xy) - H(yx) = 0 \text{ for all } x, y \in R. \tag{5.123}$$

Replacing y by yz with $z \in R$ in (5.123), we obtain

$$F(xyz) - H(yzx) = 0 \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$\begin{aligned} 0 &= F(xyz) - H(yzx) \\ &= F(xy)z + xyd(z) - H(yzx) \\ &= \big(F(xy) - H(yx)\big)z + xyd(z) + H(y)[x, z] \text{ for all } x, y, z \in R. \end{aligned}$$

Using (5.123) in the last equation, we get

$$xyd(z) + H(y)[x, z] = 0 \text{ for all } x, y, z \in R. \tag{5.124}$$

If we replace z by x in (5.124), we get

$$xyd(x) + H(y)[x,x] = xyd(x) = 0 \text{ since } [x,x] = 0 \text{ for all } x, y \in R.$$

Since R is a semiprime ring, we obtain

$$xd(x) = d(x)x = 0 \text{ for all } x \in R, \text{ since } xd(x)Rxd(x) = 0.$$

Hence we get,

$$[d(x), x] = 0 \text{ for all } x \in R. \tag{5.125}$$

If we replace x by xr with $r \in R$ in (5.124), we get the following equation. For all $x, y, z, r \in R$,

$$\begin{aligned} 0 &= xryd(z) + H(y)[xr, z] \\ &= xryd(z) + H(y)x[r,z] + H(y)[x,z]r + xyd(z)r - xyd(z)r \\ &= xryd(z) + H(y)x[r,z] - xyd(z)r + (xyd(z) + H(y)[x,z])r. \end{aligned}$$

So, using (5.124) in this equation, we find that

$$\begin{aligned} 0 &= xryd(z) + H(y)x[r,z] - xyd(z)r \\ &= x(ryd(z) - yd(z)r) + H(y)x[r,z] \\ &= x[r, yd(z)] + H(y)x[r,z] \text{ for all } x, y, z, r \in R. \end{aligned}$$

In this above equation replacing r by $d(z)$ and using (5.125), we get

$$\begin{aligned} 0 &= x[d(z), yd(z)] + H(y)x[d(z), z] \\ &= xy[d(z), d(z)] + x[d(z), y]d(z) + H(y)x[d(z), z] \\ &= x[d(z), y]d(z) \text{ since } [d(z), d(z)] = 0 \text{ for all } x, y, z \in R. \end{aligned}$$

Since R is a semiprime ring, we have

$$[d(z), y]d(z) = 0 \text{ since } [d(z), y]d(z)R[d(z), y]d(z) = \{0\} \text{ for all } x, y, z \in R. \tag{5.126}$$

Replacing y by yt with $t \in R$ in (5.126) and using (5.126), we find that

$$\begin{aligned} 0 &= [d(z), yt]d(z) \\ &= y[d(z), t]d(z) + [d(z), y]td(z) \\ &= [d(z), y]td(z) \text{ for all } y, z, t \in R. \end{aligned}$$

This yields following equation.

$$[d(z), y]t[d(z), y] = 0 \text{ for all } y, z, t \in R.$$

From the semiprimeness of R, we find that

$$[d(z), y] = 0 \text{ for all } y, z \in R. \tag{5.127}$$

Replacing x by $d(x)$ in (5.124) and using (5.127), we get, for all $x, y, z \in R$,

$$d(x)yd(z) + H(y)[d(x), z] = d(x)yd(z) = 0$$

From the semiprimeness of R, this means

$$d = 0 \text{ since } d(x)Rd(z) = \{0\}, \text{ for } x = z. \tag{5.128}$$

Hence, from the definition of F, we get

$$F(xy) = F(x)y + xd(y) = F(x)y \text{ since } d = 0 \text{ for all } x, y \in R. \tag{5.129}$$

Applying (5.128) to (5.124), we have

$$xyd(z) + H(y)[x, z] = H(y)[x, z] = 0 \text{ for all } x, y, z \in R.$$

Replacing y by yz in the last equation and using respectively (5.123) and (5.129), we get

$$F(z)y[x, z] = 0 \text{ for all } x, y, z \in R. \tag{5.130}$$

Since from (5.123) we get $F(xy) = H(yx)$, then $H(yz) = F(z)y$

If we replace x by $F(z)$ in (5.130), we obtain

$$F(z)y[F(z), z] = 0 \text{ for all } y, z \in R.$$

Hence for $y, z \in R$, we get

$$[F(z), z]y[F(z), z] = \big(F(z)z - zF(z)\big)y[F(z), z] = 0.$$

Consequently, since R is a semiprime ring, we find that $[F(z), z] = 0$, for all $z \in R$.

By the second hypothesis, we have

$$F(xy) + H(yx) = 0 \text{ for all } x, y \in R. \tag{5.131}$$

Replacing y by yz with $z \in R$ in (5.131), we obtain

$$F(xyz) + H(yzx) = 0 \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$\begin{aligned} 0 &= F(xyz) + H(yzx) \\ &= F(xy)z + xyd(z) + H(yzx) \\ &= \big(F(xy) + H(yx)\big)z + xyd(z) - H(y)[x, z] \text{ for all } x, y, z \in R. \end{aligned}$$

Using (5.131) in the last equation, we get

$$xyd(z) - H(y)[x, z] = 0 \text{ for all } x, y \in R. \tag{5.132}$$

If we replace z by x in (5.132), we get

$$xyd(x) - H(y)[x, x] = xyd(x) = 0 \text{ since } [x, x] = 0 \text{ for all } x \in R.$$

Since R is a semiprime ring, we obtain

$$xd(x) = d(x)x = 0 \text{ since } xd(x)Rxd(x) = \{0\}, \text{for } x = y \text{ for all } x \in R.$$

Hence we get,

$$[d(x), x] = 0 \text{ for all } x \in R. \tag{5.133}$$

If we replace x by xr with $r \in R$ in (5.132), we get the following equation. For all $x, y, z, r \in R$

$$\begin{aligned} 0 &= xryd(z) - H(y)[xr, z] \\ &= xryd(z) - H(y)x[r, z] - H(y)[x, z]r + xyd(z)r - xyd(z)r \\ &= xryd(z) - H(y)x[r, z] - xyd(z)r + (xyd(z) - H(y)[x, z])r. \end{aligned}$$

So, using (5.132) in this equation, we find that

$$\begin{aligned} 0 &= xryd(z) - H(y)x[r, z] - xyd(z)r \\ &= x(ryd(z) - yd(z)r) - H(y)x[r, z] \\ &= x[r, yd(z)] - H(y)x[r, z] \text{ for all } x, y, z, r \in R. \end{aligned}$$

In this equation replacing r by $d(z)$ and using (5.133), we get

$$\begin{aligned} 0 &= x[d(z), yd(z)] - H(y)x[d(z), z] \\ &= xy[d(z), d(z)] + x[d(z), y]d(z) - H(y)x[d(z), z] \\ &= x[d(z), y]d(z) \text{ since } [d(z), d(z)] = 0 \text{ for all } x, y, z \in R. \end{aligned}$$

Since R is a semiprime ring, we have

$$[d(z), y]d(z) = 0 \text{ since } [d(z), y]d(z)R[d(z), y]d(z) = 0 \text{ for all } y, z \in R. \tag{5.134}$$

Replacing y by yt with $t \in R$ in (5.134) and using (5.134), we find that

$$\begin{aligned} 0 &= [d(z), yt]d(z) \\ &= y[d(z), t]d(z) + [d(z), y]td(z) \\ &= [d(z), y]td(z) \text{ for all } x, y, z, t \in R. \end{aligned}$$

This yields following equation

$$[d(z), y]t[d(z), y] = 0 \text{ for all } y, z, t \in R.$$

From the semiprimeness of R, we find that

$$[d(z), y] = 0 \text{ for all } y, z \in R. \tag{5.135}$$

Replacing x by $d(x)$ in (5.132) and using (5.135), we get, for all $x, y, z \in R$,

$$d(x)yd(z) + H(y)[d(x), z] = d(x)yd(z) = 0.$$

From the semiprimeness of R, this means

$$d = 0, \text{ since } d(x)Rd(z) = 0, \text{ for } x = z. \tag{5.136}$$

Hence, from the definition of F, we get

$$F(xy) = F(x)y + xd(y) = F(x)y \text{ for all } x, y \in R. \tag{5.137}$$

Applying (5.136) to (5.132), we have

$$xyd(z) - H(y)[x,z] = -H(y)[x,y] = 0 \text{ for all } x,y \in R.$$

Replacing y by yz in the last equation and using respectively (5.131) and (5.137), we get

$$F(z)y[x,z] = 0 \text{ for all } x,y,z \in R. \quad (5.138)$$

Since from (5.131) we get $F(xy) = H(yx)$, then $H(yz) = F(z)y$

If we replace x by $F(z)$ in (5.138), we obtain

$$F(z)y[F(z),z] = 0 \text{ for all } x,y \in R.$$

Hence for $y,z \in R$, we get

$$[F(z),z]y[F(z),z] = \big(F(z)z - zF(z)\big)y[F(z),z] = 0.$$

Consequently, since R is a semiprime ring, we find that $[F(z),z] = 0$, for all $z \in R$.

Theorem 5.9 (Camci and Aydin, 2017) Let R be a semiprime ring, $F: R \to R$ be a multiplicative (generalized)-derivation associated with d and $H: R \to R$ be a multiplicative left centralizer. If $F(x)F(y) \mp H(xy) = 0$ holds for all $x,y \in R$, then $d = 0$. Moreover, $F(xy) = F(x)y$ holds for all $x,y \in R$ and $[F(x),x] = 0$, for all $x \in R$.

Proof. By the first hypothesis, we have

$$F(x)F(y) - H(xy) = 0 \text{ for all } x,y \in R. \quad (5.139)$$

Replacing y by yz with $z \in R$ in (5.139), we get

$$F(x)F(yz) - H(xyz) = 0 \text{ for all } x,y,z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$\begin{aligned} 0 &= F(x)F(yz) - H(xyz) \\ &= F(x)(F(y)z + yd(z)) - H(xyz) \\ &= \big(F(x)F(y) - H(xy)\big)z + F(x)yd(z) \text{ for all } x,y,z \in R. \end{aligned}$$

Using (5.139) in the last equation, we get

$$F(x)yd(z) = 0 \text{ for all } x,y \in R. \quad (5.140)$$

Replacing x by ux where $u \in R$ in (5.140) and using (5.140), from the definition of F, we obtain

$$\begin{aligned} 0 &= F(ux)yd(z) \\ &= \big(F(u)x + ud(x)\big)yd(z) \\ &= F(u)xyd(z) + ud(x)yd(z) \end{aligned}$$

$$= ud(x)yd(z) \text{ for all } x, y, z, u \in R.$$

In the last equation replacing y by $yr, r \in R$ and using that R is a semiprime ring, so we get

$$ud(x)yrd(z) = d = 0 \text{ since } ud(x)yRud(z)y = \{0\}, \text{ in particular } x = z.$$

Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$. In (5.139) replacing x by xy, we get

$$F(xy)F(y) - H(xy)y = F(x)yF(y) - H(xy)y = 0 \text{ for all } x, y \in R. \quad (5.141)$$

Multiplying (5.139) by y on the right, we have

$$F(x)F(y)y - H(xy)y = 0 \text{ for all } x, y \in R. \quad (5.142)$$

Subtracting (5.141) from (5.142), we get

$$0 = F(x)F(y)y - H(xy)y - (F(x)yF(y) - H(xy)y)$$

$$= F(x)(F(y)y - yF(y))$$

$$= F(x)[F(y), y] \text{ for all } x, y \in R.$$

Replacing x by xr with $r \in R$ in the last equation, we have

$$F(xr)[F(y), y] = F(x)r[F(y), y] = 0 \text{ for all } x, y, r \in R.$$

In this case, for $x, r \in R$, we find that

$$[F(x), x]r[F(x), x] = \big(F(x)x - xF(x)\big)r[F(x), x] = 0$$

Thus, since R is a semiprime ring, we obtain $[F(x), x] = 0$, for all $x \in R$. Then F is commuting map on R.

By the second hypothesis, we have

$$F(x)F(y) + H(xy) = 0 \text{ for all } x, y \in R. \quad (5.143)$$

Replacing y by yz with $z \in R$ in (5.143), we get

$$F(x)F(yz) + H(xyz) = 0 \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$0 = F(x)F(yz) + H(xyz)$$

$$= F(x)\big(F(y)z + yd(z)\big) + H(xyz)$$

$$= \big(F(x)F(y) + H(xy)\big)z + F(x)yd(z) = 0 \text{ for all } x, y, z \in R.$$

Using (5.143) in the last equation, we get

$$F(x)yd(z) = 0 \text{ for all } x, y, z \in R. \quad (5.144)$$

Replacing x by ux where $u \in R$ in (5.144) and using (5.144), from the definition of F, we obtain

$$\begin{aligned} 0 &= F(ux)yd(z) \\ &= \big(F(u)x + ud(x)\big)yd(z) \\ &= F(u)xyd(z) + ud(x)yd(z) \\ &= ud(x)yd(z) \ \text{ for all } x, y, z, u \in R. \end{aligned}$$

In the last equation replacing y by $yr, r \in R$ and using that R is a semiprime ring, so we get

$ud(x)yrd(z) = d = 0$ since $ud(x)yRud(z)y = \{0\}$, in particular $x = z$.

Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$. In (5.143) replacing x by xy, we get

$$F(xy)F(y) + H(xy)y = F(x)yF(y) + H(xy)y = 0 \text{ for all } x, y \in R. \tag{5.145}$$

Multiplying (5.143) by y on the right, we have

$$F(x)F(y)y + H(xy)y = 0 \text{ for all } x, y \in R. \tag{5.146}$$

Subtracting (5.145) from (5.146), we get

$$\begin{aligned} 0 &= F(x)F(y)y + H(xy)y - (F(x)yF(y) + H(xy)y) \\ &= F(x)(F(y)y - yF(y)) \\ &= F(x)[F(y), y] \ \text{ for all } x, y \in R. \end{aligned}$$

Replacing x by xr with $r \in R$ in the last equation, we have

$$F(xr)[F(y), y] = F(x)r[F(y), y] = 0 \text{ for all } x, y, r \in R.$$

In this case, for $x, r \in R$, we find that

$$[F(x), x]r[F(x), x] = \big(F(x)x - xF(x)\big)r[F(x), x] = 0$$

Thus, since R is a semiprime ring, we obtain $[F(x), x] = 0$, for all $x \in R$. Then F is commuting map on R.

The following theorems (Theorem 5.10, Theorem 5.11., Theorem 5.12., and Theorem 5.13.) gives the concepts on essentiality of semiprime ring with multiplicative (generalized)-derivation.

Theorem 5.10 (Camci and Aydin, 2017) Let R be a semiprime ring, $F: R \to R$ be a multiplicative (generalized)-derivation associated with d and $H: R \to R$ be a multiplicative left centralizer. If $F(xy) \pm H(yx) \in Z(R)$ holds for all $x, y \in R$, then $[d(x), x] = 0$, for all $x \in R$.

Proof. By the first hypothesis, we have

$$F(xy) - H(yx) \in Z(R) \text{ for all } x, y \in R. \tag{5.147}$$

If we replace y by yz with $z \in R$ in (5.147), we get

$$F(xyz) - H(yzx) \in Z(R) \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we find that

$$\begin{aligned} F(xyz) - H(yzx) &= F(xy)z + xyd(z) - H(yzx) \\ &= \big(F(xy) - H(yx)\big)z + xyd(z) + H(y)[x,z] \in Z(R) \text{ for all } x, y, z \in R. \end{aligned}$$

Commuting both sides above with z and using (5.147) we get

$$[xyd(z), z] + [H(y)[x,z], z] = 0 \text{ for all } x, y, z \in R. \quad (5.148)$$

Replacing x by xz in (5.148), we find that

$$\begin{aligned} 0 &= [xzyd(z), z] + [H(y)[xz,z], z] \\ &= [xzyd(z), z] + [H(y)(x[z,z] + [x,z]z), z] \\ &= [xzyd(z), z] + [H(y)[x,z]z, z] \\ &= [xzyd(z), z] + H(y)[x,z][z,z] + [H(y)[x,z], z]z \\ &= [xzyd(z), z] + [H(y)[x,z], z]z \text{ since, } [z,z] = 0 \text{ for all } x, y, z \in R. \quad (5.149) \end{aligned}$$

Multiplying (5.148) by z on the right, we find that

$$[xyd(z), z]z + [H(y)[x,z], z]z = 0 \text{ for all } x, y, z \in R \quad (5.150)$$

Subtracting (5.149) and (5.150) side by side, so we get

$$\begin{aligned} 0 &= [H(y)[x,z], z]z + [xzyd(z), z] - ([H(y)[x,z], z]z + [xyd(z), z]z) \\ &= [xzyd(z), z] - [xyd(z), z]z \\ &= [x([zyd(z), z] - [yd(z), z]z)] \\ &= [x(z[yd(z), z] + [z,z]yd(z) - [yd(z), z]z)] \\ &= [x(z[yd(z), z] - [yd(z), z]z)] \\ &= [x[yd(z), z], z] = 0 \text{ for all } x, y, z \in R. \end{aligned}$$

In the last equation, we replace x by rx with $r \in R$. Hence we get

$$\begin{aligned} 0 &= [rx[yd(z), z], z] \\ &= [r,z]x[yd(z), z] + r[x[yd(z), z], z] \\ &= [r,z]x[yd(z), z] \text{ for all } x, y, r, z \in R. \end{aligned}$$

In this equation, replacing r by $yd(z)$ and using that semiprimeness of R, we obtain

$$\begin{aligned} 0 &= [yd(z), z]x[yd(z), z] \\ &= [yd(z), z] \text{ since } [yd(z), z]R[yd(z), z] = \{0\} \text{ for all } y, z \in R. \end{aligned}$$

If we take $d(z)y$ instead of y and using last equation, we have

$$
\begin{aligned}
0 &= [d(z)yd(z), z] \\
&= d(z)[yd(z), z] + [d(z), z]yd(z) \\
&= [d(z), z]yd(z) \text{ for all } y, z \in R.
\end{aligned}
$$

From the last equation we have,

$$[d(z), z]y[d(z), z] = 0 \text{ for all } x, y \in R.$$

Since R is a semiprime ring, we find that $[d(z), z] = 0$, for all $z \in R$.

By the second hypothesis, we have

$$F(xy) + H(yx) \in Z(R) \text{ for all } x, y \in R. \quad (5.151)$$

If we replace y by yz with $z \in R$ in (5.151), we get

$$F(xyz) + H(yzx) \in Z(R) \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we find that

$$
\begin{aligned}
F(xyz) + H(yzx) &= F(xy)z + xyd(z) + H(yzx) \\
&= \big(F(xy) + H(yx)\big)z + xyd(z) - H(y)[x, z] \in Z(R) \text{ for all } x, y, z \in R.
\end{aligned}
$$

Commuting both sides above with z and using (5.151) we get

$$[xyd(z), z] - [H(y)[x, z], z] = 0 \text{ for all } x, y, z \in R. \quad (5.152)$$

Replacing x by xz in (5.152), we find that

$$
\begin{aligned}
0 &= [xzyd(z), z] - [H(y)[xz, z], z] \\
&= [xzyd(z), z] - ([H(y)(x[z, z] + [x, z]z), z]) \\
&= [xzyd(z), z] - [H(y)[x, z]z, z] \\
&= [xzyd(z), z] - (H(y)[x, z][z, z] + [H(y)[x, z], z]z) \\
&= [xzyd(z), z] - [H(y)[x, z], z]z = 0 \text{ since, } [z, z] = 0 \text{ for all } x, y, z \in R. \quad (5.153)
\end{aligned}
$$

Multiplying (5.152) by z on the right, we find that

$$[xyd(z), z]z - [H(y)[x, z], z]z = 0 \text{ for all } x, y, z \in R. \quad (5.154)$$

Subtracting (5.153) and (5.154) side by side, so we get

$$
\begin{aligned}
0 &= [xzyd(z), z] - [H(y)[x, z], z]z - ([xyd(z), z]z - [H(y)[x, z], z]z) \\
&= [xzyd(z), z] - [xyd(z), z]z \\
&= [x([zyd(z), z] - [yd(z), z]z)] \\
&= [x(z[yd(z), z] + [z, z]yd(z) - [yd(z), z]z)] \\
&= [x(z[yd(z), z] - [yd(z), z]z)]
\end{aligned}
$$

$$= [x[yd(z), z], z] = 0 \text{ for all } x, y, z \in R.$$

In the last equation, we replace x by rx with $r \in R$. Hence we get

$$0 = [rx[yd(z), z], z]$$
$$= [r, z]x[yd(z), z] + r[x[yd(z), z], z]$$
$$= [r, z]x[yd(z), z] \text{ for all } x, y, r, z \in R.$$

In this equation, replacing r by $yd(z)$ and using that semiprimeness of R, we obtain

$$0 = [yd(z), z]x[yd(z), z]$$
$$= [yd(z), z] \text{ since } [yd(z), z]R[yd(z), z] = \{0\} \text{ for all } y, z \in R.$$

If we take $d(z)y$ instead of y and using last equation, we have

$$0 = [d(z)yd(z), z]$$
$$= d(z)[yd(z), z] + [d(z), z]yd(z)$$
$$= [d(z), z]yd(z) \text{ for all } y, z \in R.$$

From the last equation we have,

$$[d(z), z]y[d(z), z] = 0 \text{ for all } y, z \in R.$$

Since R is a semiprime ring, we find that $[d(z), z] = 0$, for all $z \in R$. Then semiprimeness is essential.

Theorem 5.11 (Camci and Aydin, 2017) Let R be a semiprime ring, $F: R \to R$ be a multiplicative (generalized)-derivation associated with d and $H: R \to R$ be a multiplicative left centralizer. If $F(x)F(y) \pm H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[d(x), x] = 0$, for all $x \in R$.

Proof. By the first hypothesis, we have

$$F(x)F(y) - H(xy) \in Z(R) \text{ for all } x, y \in R. \tag{5.155}$$

Replacing y by yz with $z \in R$ in (5.155), we get

$$F(x)F(yz) - H(xyz) \in Z(R) \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$F(x)F(yz) - H(xyz) = F(x)\big(F(y)z + yd(z)\big) - H(xyz)$$
$$= \big(F(x)F(y) - H(xy)\big)z + F(x)yd(z) \in Z(R) \text{ for all } x, y, z \in R.$$

Commuting both side above by z and using (5.155), we get

$$[F(x)yd(z), z] = 0 \text{ for all } x, y, z \in R. \tag{5.156}$$

Replacing x by xz in (5.156) and using (5.156), hence we have

$$\begin{aligned} 0 &= [F(xz)yd(z), z] \\ &= \left[\left(F(x)z + xd(z)\right)yd(z), z\right] \\ &= [F(x)zyd(z), z] + [xd(z)yd(z), z] \\ &= [xd(z)yd(z), z] \text{ for all } x, y, z \in R. \end{aligned}$$

In the last equation, replacing x by $d(z)x$ and using this equation, we find that

$$\begin{aligned} 0 &= [d(z)xyd(z), z] \\ &= d(z)[xyd(z), z] + [d(z), z]xd(z)yd(z) \\ &= [d(z), z]xd(z)yd(z) \text{ for all } x, y, z \in R. \end{aligned}$$

This implies that

$$[d(z), z]x[d(z), z]y[d(z), z] = 0 \text{ for all } x, y, z \in R.$$

It gives that

$$(R[d(z), z])^3 = \{0\} \text{ for all } z \in R.$$

Since there is no nilpotent left ideal in semiprime rings it gives that, $R[d(z), z] = \{0\}$, $\forall z \in R$. Hence using semiprimeness of R, we conclude that $[d(z), z] = 0$, for all $z \in R$.

By the second hypothesis ,we have

$$F(x)F(y) + H(xy) \in Z(R) \text{ for all } x, y \in R. \quad (5.157)$$

Replacing y by yz with $z \in R$ in (5.157), we get

$$F(x)F(yz) + H(xyz) \in Z(R) \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we have

$$\begin{aligned} F(x)F(yz) + H(xyz) &= F(x)\left(F(y)z + yd(z)\right) + H(xy)z \\ &= \left(F(x)F(y) + H(xy)\right)z + F(x)yd(z) \in Z(R) \text{ for all } x, y, z \in R. \end{aligned}$$

Commuting above both side by z and using (5.157), we get

$$[F(x)yd(z), z] = 0 \text{ for all } x, y, z \in R. \quad (5.158)$$

Replacing x by xz in (5.158) and using (5.158), hence we have

$$\begin{aligned} 0 &= [F(xz)yd(z), z] \\ &= \left[\left(F(x)z + xd(z)\right)yd(z), z\right] \\ &= [F(x)zyd(z), z] + [xd(z)yd(z), z] \\ &= [xd(z)yd(z), z] \text{ for all } x, y, z \in R. \end{aligned}$$

In the last equation, replacing x by $d(z)x$ and using this equation, we find that

$$\begin{aligned} 0 &= [d(z)xyd(z), z] \\ &= d(z)[xyd(z), z] + [d(z), z]xd(z)yd(z) \\ &= [d(z), z]xd(z)yd(z) \text{ for all } x, y, z \in R. \end{aligned}$$

This implies that

$$[d(z), z]x[d(z), z]y[d(z), z] = 0 \quad \text{for all } x, y, z \in R.$$

It gives that,

$$(R[d(z), z])^3 = \{0\} \text{ for all } z \in R.$$

Since there is no nilpotent left ideal in semiprime rings it gives that, $R[d(z), z] = \{0\}$, $\forall z \in R$. Hence using semiprimeness of R, we conclude that, $[d(z), z] = 0$ for all $z \in R$.

Theorem 5.12 (Camci and Aydin, 2017) Let R be a semiprime ring, $F: R \to R$ be a multiplicative (generalized)-derivation associated with d and $H: R \to R$ be a multiplicative left centralizer. If $F(xy) \mp H(xy) = 0$ holds for all $x, y \in R$, then $d = 0$. Moreover $F(xy) = F(x)y$ holds for all $x, y \in R$ and $F = \pm H$.

Proof. By the first hypothesis, we have

$$F(xy) - H(xy) = 0 \text{ for all } x, y \in R.$$

So we have

$$G(xy) = 0 \quad \text{for all } x, y \in R.$$

where $G(x) = F(x) - H(x)$. Using Lemma 5.2 and Lemma 5.4, we get

$$G = 0$$

So we have

$$F = H. \tag{5.159}$$

Using the definition of F and (5.159) in the hypothesis, we get

$$\begin{aligned} 0 &= F(xy) - H(xy) \\ &= F(x)y + xd(y) - H(x)y \\ &= xd(y) \text{ for all } x, y \in R. \end{aligned}$$

Since R is a semiprime ring, we obtain

$$d = 0, \text{since } d(y)Rd(y) = \{0\}.$$

Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$.

By the second hypothesis, we have

$$F(xy) + H(xy) = 0 \text{ for all } x, y \in R.$$

So we have

$$G(xy) = 0 \quad \text{for all } x, y \in R.$$

where $G(x) = F(x) + H(x)$. Using Lemma5.3 and Lemma 5.5 we get

$$G = 0$$

So we have

$$F = -H. \tag{5.160}$$

Using the definition of F and (5.160) in the hypothesis, we get

$$0 = F(xy) + H(xy)$$

$$= F(x)y + xd(y) + H(x)y$$

$$= xd(y), \text{ for all } x, y \in R.$$

Since R is a semiprime ring, we obtain

$$d = 0, \text{ since } d(y)Rd(y) = \{0\}.$$

Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$.

Theorem 5.13 (Camci and Aydin, 2017) Let R be a semiprime ring, $F: R \to R$ be a multiplicative (generalized)-derivation associated with d and $H: R \to R$ be a multiplicative left centralizer. If $F(xy) \mp H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.

Proof. By the hypothesis, we have

$$F(xy) \mp H(xy) \in Z(R) \text{ for all } x, y \in R. \tag{5.161}$$

So we have

$$G(xy) \in Z(R) \text{ for all } x, y \in R.$$

Using Lemma 5.3 and Lemma 5.4, since if we replace y by yz with $z \in R$ in (5.161), we get

$$F(xyz) \mp H(xyz) \in Z(R) \text{ for all } x, y, z \in R.$$

Since F is a multiplicative (generalized)-derivation, we find that

$$\big(F(xy) \mp H(yx)\big)z + xyd(z) \in Z(R) \text{ for all } x, y \in R.$$

Commuting both sides above with z we get

$$[xyd(z), z] = 0 \text{ for all } x, y \in R. \tag{5.162}$$

Replacing x by xz in (5.162), we find that

$$[xzyd(z), z] = 0 \text{ for all } x, y, z \in R. \tag{5.163}$$

Multiplying (5.162) by z on the right, we find that

$$[xyd(z), z]z = 0 \quad \text{for all } x, y, z \in R. \tag{5.164}$$

Subtracting (5.163) and (5.164) side by side, so we get

$$0 = [xzyd(z), z] - [xyd(z), z]z$$
$$= [x([zyd(z), z] - [yd(z), z]z)]$$
$$= [x(z[yd(z), z] + [z, z]yd(z) - [yd(z), z]z)]$$
$$= [x(z[yd(z), z] - [yd(z), z]z)]$$
$$= [x[yd(z), z], z] = 0, \forall x, y, z \in R.$$

In the last equation, we replace x by rx with $r \in R$. Hence we get

$$0 = [rx[yd(z), z], z]$$
$$= r[x[yd(z), z], z] + [r, z]x[yd(z), z]$$
$$= [r, z]x[yd(z), z] \text{ for all } x, y, r, z \in R.$$

In this equation, replacing r by $yd(z)$ and using that semiprimeness of R, we obtain

$$0 = [yd(z), z]x[yd(z), z]$$
$$= [yd(z), z]R[yd(z), z]$$
$$= [yd(z), z] \text{ for all } y, z \in R.$$

If we take $d(z)y$ instead of y and using last equation, we have

$$0 = [d(z)yd(z), z]$$
$$= d(z)[yd(z), z] + [d(z), z]yd(z)$$
$$= [d(z), z]yd(z) \text{ for all } y, z \in R.$$

From the last equation we have,

$$[d(z), z]y[d(z), z] = 0 \text{ for all } y, z \in R.$$

Since R is a semiprime ring, we find that

$$[d(z), z] = 0, \text{ since } [d(z), z]R[d(z), z] = \{0\} \text{for all } z \in R.$$

The following examples show that the hypothesis of semiprimeness is essential.

Example 5.1 (Camci and Aydin, 2017) Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers and the maps $F,d,H\colon R \to R$ defined by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & \lambda b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda a^2 & \lambda b^2 \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix}, H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \lambda a & \lambda b \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } \lambda \in \mathbb{Z}.$$

Since $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \{0\}, R$ is not semiprime. Moreover, it is easy to show that, F is a multiplicative (generalized)-derivation associated with d and $H(xy) = H(x)y = 0$ holds for all $x,y \in R$. But, we observe that $d(R) \neq \{0\}$ and $F(xy) \neq F(x)y$ for $x,y \in R$. Hence the semiprimeness hypothesis in the theorem 5.12 is crucial.

Example 5.2 (Camci and Aydin, 2017) Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers and the maps $F,d,H\colon R \to R$ defined by

$$F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda a & 0 \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{pmatrix},\ d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda ab & \lambda b^2 \\ 0 & 0 & -\lambda c \\ 0 & 0 & 0 \end{pmatrix}, H\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \lambda^2 a & \lambda^2 b \\ 0 & 0 & \lambda^2 c \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } \lambda \in \mathbb{Z}.$$

Then R is not semiprime and it is easy to show that, F is a multiplicative (generalized)-derivation associated with d and $H(xy) = H(x)y, F(x)F(y) - H(xy) = 0$ holds for all $x,y \in R$. We observe that $d(R) \neq \{0\}$ and $F(xy) \neq F(x)y$ for $x,y \in R$. Hence the semiprimeness hypothesis in the theorem 5.9 is essential.

Example 5.3 (Camci and Aydin, 2017) Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers and the maps $F,d,H\colon R \to R$ defined by

$F\begin{pmatrix}0 & 0 & 0\\ a & 0 & 0\\ b & c & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ a^2 & 0 & 0\\ b+c & 0 & 0\end{pmatrix}$, $d\begin{pmatrix}0 & 0 & 0\\ a & 0 & 0\\ b & c & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ a & 0 & 0\\ b^2 & 0 & 0\end{pmatrix}$, and

$H\begin{pmatrix}0 & 0 & 0\\ a & 0 & 0\\ b & c & 0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ ab & 0 & 0\end{pmatrix}$, where $\lambda \in \mathbb{Z}$.

Since $\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} R \begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=\{0\}$, R is not semiprime. It yields that, F is multiplicative (generalized)-derivation associated with d and $H(xy)=H(x)y, F(x)F(y)-H(xy)=0$ holds for all $x, y \in R$. But, we see that $d(R)\neq\{0\}$ and $F(xy)\neq F(x)y$ for $x, y \in R$. Hence the semiprimeness hypothesis in the theorem 5.9 is essential.

6. SUMMARY AND CONCLUSION

6.1. Summary

In general, the main objective of this project to study functional identities on semiprime ring with multiplicative (generalized)-derivation. A ring R is said to be semiprime if for any $a \in R$ such that $aRa = \{0\}$ implies $a = 0$ and an additive mapping d from R to R is said to be derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A map $F: R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation associated to a map $d: R \to R$ if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$ where d is any map on R (not necessarily a derivation nor additive map). Preliminary concepts, definitions, examples, lemmas and theorems were presented to make concept clear. We are used following functional identities on semiprime rings with multiplicative (generalized)-derivations to prove the theorems for all $x, y \in L$:

(i) $F(xy) \pm xy = 0$

(ii) $F(x)F(y) \pm xy = 0$

(iii) $F(xy) \pm yx \in Z(R)$

(iv) $F(xy) \pm H(yx) = 0$

(v) $F(xy) \pm H(xy) \in Z(R)$.

Most of the results presented in chapter 5 give concepts on multiplicative (generalized)-derivations is commuting map on ring R and essentiality of semiprime ring in multiplicative (generalized)-derivations. It is proved that different theorems which facilitates our aim of this project to clear. Finally, some examples are given to shows that the semiprimeness hypothesis are necessary and do not hold for arbitrary rings.

6.2. Conclusion

In this project, functional identities on semiprime ring with multiplicative (generalized)-derivations have been studied. Overall finding of this project work was show essentiality of semiprime rings with multiplicative (generalized)-derivations. Moreover, from this project we can concluded that multiplicative (generalized)-derivation is commuting map on ring R in Theorem 5.1, Theorem 5.2, Theorem 5.3, Theorem 5.8., and Theorem 5.9. In Theorem 5.4, Theorem 5.5., Theorem 5.6., Theorem 5.7, Theorem 5.10., Theorem 5.11., Theorem 5.12., and Theorem 5.13 we showed that semiprimeness is necessary. Finally, we shown that hypothesis of semiprimeness is essential and do not hold for arbitrary rings in Theorem 5.9 and Theorem 5.12.

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