SEMIGROUP IDEALS AND COMMUTATIVITY OF NEAR RINGS WITH SEMIDERIVATIONS SATISFYING CERTAIN DIFFERENTIAL IDENTITIES

MSc PROJECT

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Semigroup Ideals and Commutativity of Near Rings with Semiderivations Satisfying Certain Differential Identities

**A Project Submitted to the Department of Mathematics,
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MASTER OF SCIENCE IN MATHEMATICS
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I hereby certify that I have read and evaluated this Project titled '*Semigroup Ideals and Commutativity of Near Rings with Semiderivations Satisfying Certain Differential Identities*' prepared under my guidance by Leta Hailu. I recommend that it be submitted as fulfilling the project requirement.

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As member of the board of Examiners of the MSc Project Open Defense Examination, we certify that we have read and evaluated the Project prepared by Leta Hailu and examined the candidate. We recommend that the Project be accepted as fulfilling the Project requirement for the degree of *Master of Science in Mathematics* (Algebra).

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DEDICATION

I dedicate this project work to my beloved families.

STATEMENT OF THE AUTHOR

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BIOGRAPHICAL SKETCH

The author was born in 1977 (G.C) on August 1 in Oromia Regional State, Bale Zone, Adaba Wereda, Furuna Kebele.

He attended his primary education at Furuna primary schools. Then after, he attended his junior and secondary education at Adaba secondary schools. He joined Jimma Teachers Training College in 1996 and received diploma in Mathematics in July 1998. He then joined Haramaya University in 2001 and received Bachelor of Education degree in Mathematics in October 2005.

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Semigroup Ideals and Commutativity of Near Rings with Semiderivations Satisfying Certain Differential Identities

ABSTRACT

The main purpose of this project is to elaborate the ideas and concepts of semigroup ideals and commutativity of near rings with semiderivations satisfying certain differential identities. The important preliminary concepts, definitions, lemmas and theorems were elaborated with detailed explanatory steps to make the concept clear. In addition, to study commutativity of near rings definitions and properties of semiderivation and semigroup ideals were employed. Detailed explanatory steps on the proofs of the theorems and lemmas on commutativity of near rings admitting a semiderivation satisfying certain differential identities were provided. Also, we have seen the cases that in some results the primeness, n*-torsion free and zero symmetric of near ring* N *in the setting of semigroup ideals involving semiderivations are sufficient condition for commutativity of near ring. Further in the main results of this study detailed proofs by including more explanatory steps to show commutativity of near ring* N *under the assumptions "a* 2*-torsion free prime near ring* N *that admits a nonzero semiderivation* f *satisfying* $f(N)$ *is contained in the multiplicative center of* $Z(N)$*" and "a prime near ring* N *with semiderivation* f *acting as a homomorphism or anti-homomorphism in the setting of a semigroup ideal* U *of* N*."*

Key words: Prime near rings, Semiderivations, Commutativity.

1. INTRODUCTION

1.1. Background of the Study

The concept of near ring began as an axiomatic theory by Dickson in 1905 (Pilz,1983). Since then the theory of near ring has been developed much and at present it becomes a sophisticated theory with numerous areas. A near ring is exactly what is needed to describe the structure of the endomorphisms of various mathematical structures adequately. The notion of near ring was introduced as a nonempty set N with two binary operations usually called addition and multiplication, which is a group (not necessarily abelian) with the first operation, semigroup with second operation and only satisfies one side distributive properties (Pilz,1983). Every ring is a near ring. It is natural to generalize various concepts of rings to near rings.

The notion of ring with derivation is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. The study of derivations in rings though initiated long back, but got impetus only after Posner (1957) established two very striking results on derivations in prime rings. The result states that; in a 2-torsion-free prime ring, if the iterate of two derivations is a derivation, then one of them must be zero; and a prime ring R admitting a nonzero centralizing derivation d must be commutative. Also, there has been considerable interest in investigating commutativity of rings.

Bergen (1983) introduced the notion of semiderivation in ring as: A function $f: R \rightarrow R$ is said to be a semiderivation of a ring R if there exists a function $g: R \rightarrow R$ such that

$$\left.\begin{array}{c} f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y) \text{ for all } x, y \in R \\ \text{and} \\ f\big(g(x)\big) = g\big(f(x)\big) \text{ for all } x, y \in R \end{array}\right\} \quad (1.1)$$

He also examined the relationship between the structure of R and that of $d(R)$ if R is a prime ring and $d \neq 0$ is a derivation of R.

Chang (1984) studied on semiderivations of prime rings. He obtained some results on derivations of prime rings to prime rings with semiderivations.

Bell and Mason (1987) introduced the notion of derivation in a near ring N as follows:
A derivation on N is defined to be an additive endomorphism satisfying the product rule

$$d(xy) = xd(y) + d(x)y \text{ for all } x, y \in N.$$

They extend many results of derivation in rings to derivation in near rings. It was shown by Wang (1994) that

$$d(xy) = xd(y) + d(x)y \text{ if and only if } d(xy) = d(x)y + xd(y) \text{ for all } x, y \in N.$$

Bell and Martindale (1988) motivated by the notion of indicated by Bergen (1983) introduced a new finding on semiderivations and commutativity in prime rings which is imposed on an ideal of a ring R rather than on R itself. They proved that if R is a prime ring, U is a nonzero ideal of R and f is a nonzero semiderivation on R, then f is nonzero on U. Consequently, many authors generalize the concepts of semiderivations in rings in different directions.

Motivated by the definition of semiderivation in rings given by Bergen (1983), Boua and Oukhtite (2013) introduced the notion of semiderivation in near rings as follows: An additive mapping $f: N \to N$ is called semiderivation if there is an additive mapping $g: N \to N$ such that (1.1) holds true for N.
In case g is the identity map on N, f is of course just a derivation on N, so the notion of semiderivation generalizes that of derivation.

Bell and Mason (1987) proved partially distributive law for derivation on near rings and extended some theorem of derivation on rings to derivation on near rings which facilitate near rings are commutative rings. Boua and Oukhtite (2013) extended these results for semiderivat ions on near rings.

Ashraf and Boua (2016) proved some differential identities on near rings involving semiderivations which show that near rings are commutative rings. By the same year Kim (2016) extends some properties of prime rings with derivation to prime near rings with semiderivations.

Bell (1997) studied semigroup ideal of a near ring according to his definition, a nonempty subset U of N is said to be a semigroup right ideal (semigroup left ideal) if $UN \subseteq U$ ($NU \subseteq U$); and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. He extended some results on derivations of prime near rings to derivations on prime near rings in the setting of a semigroup ideal.

Boua *et al*. (2014) proved some identities on near ring in the setting of semigroup ideals involving a semiderivation. Consequently, Ali *et al*. (2016) extended some theorem in the setting of semigroup ideals of prime near rings admitting a nonzero semiderivation, thereby extending some known results on derivations. Also, they studied semiderivation acting as a homomorphism or anti-homomorphism in the setting of semigroup ideal U of N.

In this project work we elaborate the ideas and concepts of near ring involving semiderivation to be commutative and to discuss the preliminary ideas and concepts on semigroup ideals and commutativity of near rings involving semiderivations satisfying certain differential identities. Also, provide more details on the proofs of the theorems and lemmas on commutativity of near rings admitting a semiderivation satisfying certain differential identities.

1.2. Statement of the Problem

A near ring is a generalization of rings. The notion of semiderivation generalizes that of derivation. A typical problem in near ring N with semiderivation is to state and prove theorems that deal with sufficient conditions under which N is commutative ring.
In a connection with this many results have been stated and proved in the last five years.

Boua and Oukhtite (2013) stated and proved that if N is a 2-torsion free prime near ring admitting a nonzero semiderivation f with

(i) $f(N) \subseteq Z(N)$

(ii) $f([x,y]) = 0$ for all $x, y \in N$

then N is commutative ring.
Boua *et al*. (2014) stated and proved that if N is a 2-torsion free prime near ring and U is nonzero semigroup ideal of N admitting a nonzero semiderivation f with

(i) $f(U) \subseteq Z(N)$

(ii) $f(-U) \subseteq Z(N)$

(iii) $f([x,y]) = 0$ for all $x, y \in U$

(iv) $f([x,y]) = [x,y]$ for all $x, y \in U$

(v) $f(xoy) = 0$ for all $x, y \in U$

(vi) $f(xoy) = xoy$ for all $x, y \in U$

then N is commutative ring.
More results on commutativity of near rings satisfying certain differential identities involving semiderivations can be found in Kim (2016).
This project tried to:

i. Elaborate the ideas and concepts on semigroup ideals and commutativity of near rings involving semiderivations satisfying certain differential identities presented in Boua and Oukhtite (2013), Boua *et al*. (2014), Kim (2016) and Ali *et al*. (2016).
ii. Provide more detailed proofs of the theorems and lemmas stated and proved by Boua and Oukhtite (2013), Boua *et al*. (2014), Kim (2016) and Ali *et al*. (2016).

1.3. Objectives of the Study

The main objective of this project was to elaborate the ideas and concepts on semigroup ideals and commutativity of near rings satisfying certain differential identities involving semiderivati ons.

The study explored the following specific objectives:

- To discuss the preliminary ideas and concepts on semigroup ideals and commutativity of near rings involving semiderivations satisfying certain differential identities.
- To elaborate the ideas and concepts of near ring involving semiderivation to be commutative.
- To provide more details on the proofs of the theorems and lemmas on commutativity of near rings admitting a semiderivation satisfying certain differential identities.
- To elaborate the ideas and concepts on commutativity of a near ring involving semiderivation in the setting of semigroup ideals.

2. LITERATURE REVIEW

Near rings are natural objects of study as algebraic objects, relating to many other abstract algebraic entities. Many interesting structural results are brought out, and the genealogy of the results is made quite clear (Pilz,1983).

Bell and Mason (1987) proved partial distributive law for near ring and they extended some results for derivation in rings to derivation in near rings as, if N is a 2-torsion free prime near ring admitting a derivation such that

(i) $d(N) \subseteq Z(N)$

(ii) $[d(x), d(y)] = 0$ for all $x, y \in N$

then N is commutative ring.

Motivated by these results Boua and Oukhtite (2013) extended partial distributive law results for semiderivations in near rings and they extended some results of derivation on near ring to semiderivation on near ring as a 2-torsion free prime near ring admitting a nonzero semiderivation for which $f(N) \subseteq Z(N)$ and $f([x, y]) = 0$, then N is commutative ring.

Thereafter, Kim (2016) extended some results of derivation on near ring to semiderivation in near rings and proved the following identities as

(i) $f(xoy) = xoy, f(xoy) = [x, y]$

(ii) $f[x, y] = xoy$ for all $x, y \in N$ which showed that N is commutative ring.

Bell (1997) extended some results of derivation on near rings in the setting of semigroup ideal U of N as $d(U) \subseteq Z(N)$ and $[d(U), d(U)] = \{0\}$ which facilitates near ring is commutative.

Motivated by these results Boua *et al.* (2014) proved some differential identities on near rings in the setting of semigroup ideals involving semiderivation as

(i) $f(U) \subseteq Z(N)$,

(ii) $f([x, y]) = [x, y]$ for all $x, y \in U$

(iii) $f(xoy) = xoy$ etc. for all $x, y \in U$

then N is commutative ring.

Ali *et al*. (2016) proved some theorems of nonzero semiderivation of near rings admitting a nonzero semigroup ideal U of N, thereby they extended some known results on derivations to semiderivation on near ring as if N is a 2-torsion free zero-symmetric prime near ring, U be a nonzero semigroup ideal of N and f is a nonzero semiderivation of N associated with a map g such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$ such that $f(U) \subseteq Z(N)$ and $[f(U), f(U)] = \{0\}$, then N is commutative ring.
In addition, they proved if N is a 2-torsion free zero-symmetric prime near ring, U be a nonzero semigroup ideal of N and f is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$, then $f^2(U) \neq \{0\}$.

Bell and Kappe (1989) proved that if R is a semiprime ring and d is a derivation on R which is either an endomorphism or an anti-endomorphism on R, then $d = 0$.
Motivated by these results Ali *et al*. (2016) established similar results in the setting of a semigroup ideal of a prime near ring admitting a semiderivation f associated with a map g such that $g(uv) = g(u)g(v)$, for all $u, v \in U$ which acts as a homomorphism or an anti-homomorphism on U, then either f is the identity map on U or $f = 0$.
Consequently, Ali *et al*. (2016) proved similar results in the setting of a semigroup ideal of a prime near ring admitting a semiderivation f associated with a map g such that $g(uv) = g(u)g(v)$, for all $u, v \in U$ which acts as an anti-homomorphism on U, then N is commutative ring.

Generally, in this project work the concept of semiderivations in near rings through the study of structure and commutativity of near rings admitting semiderivations satisfying certain differential identities and the commutativity conditions of a near ring involving semiderivation in the setting of semigroup ideal U of a near ring N was elaborated.

3. MATERIALS AND METHODS

Sources in the web and libraries were used to collect all the pieces of information about the semigroup ideals and commutativity of near rings with semiderivations and recorded subsequently.

The overall procedure of this project was as follows:

- Relevant journals and books were consulted to gather information about semigroup ideals and commutativity of near rings with semiderivations satisfying certain differential identities.
- The collected information was analyzed.
- Important preliminary concepts, definitions, and theorems were elaborated to make the concept clear.
- Proved some theorems on semigroup ideals and commutativity of near rings with semiderivations satisfying certain differential identities with detailed explanatory steps to make the concept clear.
- Important lemmas were used to prove the theorems.

4. PRELIMINARIES

In this chapter, we deal with basic definitions, examples and lemmas which have been important ideas and concepts for the main results of chapter 5.

4.1. Basic Concepts of Near Rings

We begin with the following definitions, examples and lemmas, which is essential for elaborating the proof of the main results.

Definition 4.1.1. A ring $(R, +, \cdot)$ is a set R together with two binary operation $"+"$ and $"\cdot"$ which we call addition and multiplication, defined on R such that the following axioms are satisfied:

(i) $(R, +)$ is abelian group

(ii) $(R, \cdot)$ is a semi group

(iii) For all $a, b, c \in R$ the left distributive law $a \cdot (b + c) = a \cdot b + a \cdot c$ and the right distributive law $(a + b) \cdot c = a \cdot c + b \cdot c$ hold.

Definition 4.1.2. (Pilz,1983) A near ring $(N, +, \cdot)$ is an algebraic structure together with two binary operations addition $"+"$ and multiplication $"\cdot"$ such that

(i) $(N, +)$ is a group not necessarily abelian

(ii) $(N, \cdot)$ is a semi group

(iii) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$ (left distributive law)

Hence, this type of near ring will be termed as left near ring.

Remark 4.1.3. Similarly one can define right near ring by redefining condition (iii) in definition 4.1.2 $(a + b). c = a \cdot c + b \cdot c$ for all $a, b, c \in N$. In this project work a near ring mean that only a left near ring unless otherwise stated.

Remark 4.1.4. (Kandasamy, 2002) Every ring is a near ring but the converse is not true in general.

Example 4.1.5. (Kandasamy, 2002) Let $\mathbb{Z}$ be the set of integers. Now define multiplication $"\cdot"$ on $\mathbb{Z}$ by $a \cdot b = b$ for all $a, b \in \mathbb{Z}$.

(i) Clearly $(\mathbb{Z}, +)$ is an abelian group

(ii) $(\mathbb{Z}, \cdot)$ is a semi group

(iii) $a \cdot (b + c) = b + c$ (by definition)

On the other side $ab + ac = b + c$ for all $a, b, c \in \mathbb{Z}$.

Therefore, $(\mathbb{Z}, +, \cdot)$ is a left near ring.

Definition 4.1.6. (i) A ring $(R, +, \cdot)$ is a commutative ring if and only if $ab = ba$ for all $a, b \in R$.

(ii) A ring $(R, +, \cdot)$ is said to have a unity if and only if there is an element $1 \in R$ with $1 \neq 0$ and $1. a = a = a. 1$ for all $a \in R/\{0\}$. Such ring is called a ring with unity.

Remark 4.1.7. Let N be a near ring. Then,

(i) If $(N, +)$ is abelian, then we call N an abelian near ring.

(ii) If $(N, \cdot)$ is commutative, then we call N a commutative near ring.

Definition 4.1.8. (Bell and Mason, 1987) Let N be a near ring. Then an additive mapping $d: N \to N$ is said to be a derivation if

$$d(xy) = xd(y) + d(x)y \text{ for all } x, y \in N.$$

Note: Additive mapping $d: N \to N$ implies $d(x + y) = d(x) + d(y)$. Also, this holds true for semiderivation.

Definition 4.1.9. An element a of a near ring N is called

(i) Left zero divisor if $ab = 0$ for some nonzero element $b \in N$.

(ii) Right zero divisor if $ba = 0$ for some nonzero element $b \in N$.

In particular, a is zero divisor if it is a left or right zero divisor.

Definition 4.1.10. (Boua *et al.*, 2014 and Ali *et al.*, 2016) For any pair of elements x, y in a near ring N, the additive commutator is denoted by (x, y) and is defined as $x + y - x - y$ and multiplicative commutator is denoted by $[x, y]$ and is defined as $xy - yx$ while the multiplicative anti-commutator denoted by xoy and is defined as $xy + yx$ for all $x, y \in N$.

Remark 4.1.11. Let N be a near ring. Then the following basic commutator identities hold for any $x, y, z \in N$.

$$(i)\ [xy, z] = x[y, z] + [x, z]y$$
$$(ii)\ [x, yz] = y[x, z] + [x, y]z$$
$$(iii)\ x\ o\ (yz) = (x\ o\ y)z - y[x, z] = y(x\ o\ z) + [x, y]z$$
$$(iv)\ (xy)\ o\ z = x(y\ o\ z) - [x, z]y = (x\ o\ z)y + x[y, z]$$

Definition 4.1.12. (Boua *et al.*, 2014) Multiplicative Center of a near ring N is the set of all those elements of N which commute with every elements of N under multiplication and is denoted by $Z(N)$,

that is,

$$Z(N) = \{x \in N\ /\ xy = yx \text{ for all } y \in N\}$$

Definition 4.1.13. (Boua and Oukhtite, 2013) A near ring N is said to be prime if for any $x, y \in N$, $xNy = \{0\}$ implies $x = 0$ or $y = 0$.

Definition 4.1.14. (Boua and Oukhtite, 2013) An element x in a near ring N is said to be n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in N$. Now, a near ring N is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in N$.

Definition 4.1.15. (Boua and Oukhtite, 2013) A near ring N is called zero symmetric if $x0 = 0x = 0$ for all $x \in N$.

In this project work N denotes zero symmetric left near ring unless otherwise mentioned.

The following lemmas $(4.1.16 - 4.1.18)$ are Bell results which we have been used for the part of main results.

Lemma 4.1.16. (Bell,1997) Let N be a prime near ring. If $x \in Z(N) \setminus \{0\}$, then x is not a zero divisor.

Proof: Let $x \in Z(N) \setminus \{0\}$ and for any $b \in N$, $xb = 0$, then $xcb = 0$ for all $b, c \in N$.

Thus, we get

$$xNb = \{0\} \text{ for all } c \in N$$

Since N is prime near ring and $x \neq 0$, then $xNb = \{0\}$ implies $b = 0$.

Thus, x is not a zero divisor.

Lemma 4.1.17. (Bell,1997) Let N be a prime near ring. If $Z(N)\backslash\{0\}$ contains an element z for which $z + z \in Z(N)$, then $(N, +)$ is abelian.

Lemma 4.1.18. (Bell,1997) Let N be a prime near-ring. If $z \in Z(N)\backslash\{0\}$ and $xz \in Z(N)$ or $zx \in Z(N)$, then $x \in Z(N)$.

Proof: Assume that $xz \in Z(N)$, then $[xz, y] = 0$ for all $y \in N$

Since $z \in Z(N)\backslash\{0\}$, it follows that

$[x, y]z = 0$ since $z \in Z(N)\backslash\{0\}$

$[x, y] = 0$ since $z \in Z(N)\backslash\{0\}$, then z is not a zero divisor

$xy = yx$ for all $y \in N$

Hence, $x \in Z(N)$.

4.2. Semigroup Ideals and Semiderivation of Near Rings

We use the following definitions and lemmas, which is essential for elaborating the proof of the main results.

Definition 4.2.1. (Boua and Oukhtite, 2013) Let N be a near ring. Then an additive mapping $f: N \to N$ is called semiderivation if there is an additive mapping $g: N \to N$ such that

$$f(xy) = xf(y) + f(x)g(y) = g(x)f(y) + f(x)y$$

and

$$f\big(g(x)\big) = g\big(f(x)\big) \text{ for all } x, y \in N.$$

Remark 4.2.2. (Boua and Oukhtite, 2013) Every derivation is a semiderivation on a near ring N.

Proof: Let a derivation d be on N then $d(xy) = xd(y) + d(x)y$

Now let g be on N such that $g: N \to N$ be defined as $g(x) = x$ for all $x \in N$

then $d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y$ for all $x, y \in N$

Again $d\big(g(x)\big) = d(x)$ and $g\big(d(x)\big) = d(x)$ for all $x \in N$

Thus, d is a semiderivation on N.

Remark 4.2.3. (Boua and Oukhtite, 2013) Every semiderivation may not be a derivation. For this we have an example:

Example 4.2.4. (Ali *et al.*, 2016) Let $N = N_1 \oplus N_2$, where N_1 is a zero symmetric near ring and N_2 is a ring. Then the map $f: N \to N$ defined by

$$f\big((x, y)\big) = (0, y)$$

is a semiderivation associated with a function $g: N \to N$ such that $g(x, y) = (x, 0)$.

However, f is not a derivation on N.

To see this:

Let $x = (x_1, x_2) \in N$

$y = (y_1, y_2) \in N$, where $x_1, y_1 \in N_1$ and $x_2, y_2 \in N_2$

Now let us check the three properties of semiderivations.

(i) $f(x + y) = f\big((x_1, x_2) + (y_1, y_2)\big) = f(x_1 + y_1, x_2 + y_2)$

$= (0, x_2 + y_2)$

$= (0, x_2) + (0, y_2)$

$= f(x_1, x_2) + f(y_1, y_2)$

$= f(x) + f(y)$ for all $x, y \in N$

(ii) Suppose $f(xy) = f\big((x_1, x_2).(y_1, y_2)\big) = f(x_1y_1, x_2y_2) = (0, x_2y_2)$

and also $f(x)g(y) + xf(y) = f\big((x_1, x_2)\big)g(y_1, y_2) + (x_1, x_2)f(y_1, y_2)$

$= (0, x_2)(y_1, 0) + (x_1, x_2)(0, y_2)$

$= (0, x_2y_2)$

(iii) Suppose $f\big(g(x)\big) = f\big(g(x_1, x_2)\big) = f(x_1, 0) = (0,0)$

again $g\big(f(x)\big) = g\big(f(x_1, x_2)\big) = g(0, x_2) = (0,0)$

Hence, using $(i), (ii)$ and (iii) f is a semiderivation on N.

Now let us check for derivation on N

$$f(xy) = xf(y) + f(x)y$$

$$f\big((x_1, x_2)(y_1, y_2)\big) = (x_1, x_2)f(y_1, y_2) + f(x_1, x_2)(y_1, y_2)$$

$$f(x_1y_1, x_2y_2) = (x_1, x_2)(0, y_2) + (0, x_2)(y_1, y_2)$$

$$(0, x_2y_2) \neq (0, 2x_2y_2)$$

Thus, f is not a derivation on N.

Lemma 4.2.5. (Boua and Oukhtite, 2013) Let f be a semiderivation of a prime near-ring N and $a \in N$ such that $af(x) = 0$ for all $x \in N$, then $a = 0$ or $f = 0$.

Proof: Assume that $af(x) = 0$ for all $x \in N$

$af(xy) = 0$ for all $y \in N$

$a[f(x)g(y) + xf(y)] = 0$ (definition of semiderivation)

$af(x)g(y) + axf(y) = 0$

$0. g(y) + axf(y) = 0$

$axf(y) = 0$ for all $x, y \in N$

$aNf(y) = \{0\}$

$a = 0$ or $f(y) = 0$

Hence, $a = 0$ or $f = 0$.

Lemma 4.2.6. (Kim, 2016) Let N be a prime near-ring and $a \in N$. If f is a nonzero semiderivation associated with a surjective function $g: N \to N$. If $af(N) = 0$, then a $= 0$.

Proof: Suppose that $af(N) = 0$ and $x \in N$. Since $f \neq 0$, there is $y \in N$ such that

$f(y) \neq 0$, and $af(xy) = 0$

Hence we obtain

$$a(f(x)g(y) + xf(y)) = af(x)g(y) + axf(y) = 0$$

Since $af(x) = 0$, we have $axf(y) = 0$ for every $x \in N$. Since N is prime and $f(y) \neq 0$, we get a $= 0$.

Lemma 4.2.7. (Boua and Oukhtite, 2013) Let f be a semiderivation of a near ring N. Then N satisfies the following partial distributive law:

$$(xf(y) + f(x)g(y))g(z) = xf(y)g(z) + f(x)g(yz) \text{ for all } x, y, z \in N.$$

Proof: Let $x, y, z \in N$, then by the definition of semiderivation we have

$$\begin{aligned} f((xy)z) &= xyf(z) + f(xy)g(z) \\ &= xyf(z) + (xf(y) + f(x)g(y))g(z). \qquad (4.1) \end{aligned}$$

On the other hand,

$$\begin{aligned} f(x(yz)) &= xf(yz) + f(x)g(yz) \\ &= x(yf(z) + f(y)g(z)) + f(x)g(yz) \end{aligned}$$

$$= xyf(z) + xf(y)g(z) + f(x)g(yz). \qquad (4.2)$$

Comparing equations (4.1) with (4.2), we get

$$(xf(y) + f(x)g(y))g(z) = xf(y)g(z) + f(x)g(yz) \text{ for all } x, y, z \in N.$$

Lemma 4.2.8. (Boua *et al.*, 2014) Let N be a near-ring and f is a semiderivation of N associated with an automorphism g. Then N satisfies the following partial distributive law.

(i) $(f(x)y + g(x)f(y))z = f(x)yz + g(x)f(y)z$ for all $x, y, z \in N$.

(ii) $(xf(y) + f(x)g(y))z = xf(y)z + f(x)g(y)z$ for all $x, y, z \in N$.

Definition 4.2.9. (Bell, 1997) A nonempty subset U of a near ring N is said to be right (left) semigroup ideal of N if $UN \subseteq U$ $(NU \subseteq U)$, U is said to be a semigroup ideal if it is both a right semigroup ideal and a left semigroup ideal of N.

Example 4.2.10. Let $N = \{0, a, b, c\}$ with addition and multiplication tables defined as below:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

If we take $A = \{0, a\}$, $B = \{0, a, b\}$ and $C = \{0, a, c\}$ then B, C are semigroup right ideals of N and A is a semigroup ideal of N.

Definition 4.2.11. (Kandasamy, 2002) Let $(N_1, \oplus, *)$ and $(N_2, +, .)$ be two near rings. Then a mapping $f: N_1 \to N_2$ is called a near ring homomorphism if

(i) $f(r_1 \oplus r_2) = f(r_1) + f(r_2)$

(ii) $f(r_1 * r_2) = f(r_1) \,.\, f(r_2)$ for all $r_1, r_2 \in N_1$

Definition 4.2.12. (Kandasamy,2002) Let $(N_1, \oplus, *)$ and $(N_2, +, .)$ be two near rings. Then a mapping $f: N_1 \to N_2$ is called a near ring anti-homomorphism if

(i) $f(s_1 \oplus s_2) = f(s_1) + f(s_2)$

(ii) $f(s_1 * s_2) = f(s_2) \,.\, f(s_1)$ for all $s_1, s_2 \in N_1$

Lemma 4.2.13. (Bell,1997) Let N be a prime near ring and U be a nonzero semigroup ideal of N.

(i) If $x \in N$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.

(ii) If $x \in N$ centralizes U, then $x \in Z(N)$.

Proof: (i) Suppose $xU = \{0\}$

For any $u \in U, b \in N, ub \in U$ and $bu \in U$, then

$xbu = 0$ for all $b \in N$

$xNu = \{0\}$

Since N is prime near ring $x = 0$ or $u = 0$ but $u \neq 0$, then $x = 0$.

(ii) Since x centralizes U, $[x, u] = 0$ for all $u \in U$, $[x, U] = \{0\}$

For any $u \in U, y \in N, uy \in U$ and $yu \in U$

$[x, uy] = 0$ for all $y \in N, u \in U$

$u[x, y] + [x, u]y = 0$

$u[x, y] = 0$ for all $y \in N$

By using (i) $[x, y] = 0$ for all $y \in N$.

$xy = yx$ for all $y \in N$

Hence, $x \in Z(N)$.

Lemma 4.2.14. (Bell,1997) Let N be a prime near-ring, and U a nonzero semigroup ideal of N. If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

Proof: Suppose $xUy = \{0\}$ for all $x, y \in N$

For any $u \in U, b \in N$ we have $ub \in U$

Now by substituting ub in U we get, $xuby = 0$

$xuNy = \{0\}$ for all $b \in N$

Since N is a prime near-ring, then we have $xu = 0$ or $y = 0$.

But from $xu = 0$, we get $x = 0$ by Lemma 4.2.13(i).

Hence, $x = 0$ or $y = 0$.

Lemma 4.2.15. (Bell,1997) Let N be a prime near ring and $Z(N)$ contains a nonzero semigroup left ideal or semigroup right ideal, then N is a commutative ring.

Proof: By the hypothesis $U \subseteq Z(N)$

Suppose U is nonzero semigroup left ideal of N.

Then, $Ux = xU$ for all $x \in N$ by hypothesis (4.3)

Since $ru \in U$ for all $r \in N, u \in U$, then

$ur \in U$ since $ur = ru$ for all $r \in N, u \in U$ by (4.3)

Now U is nonzero semigroup right ideal of N.

Thus, for any $y \in N$, $Uy = yU$

Therefore, U is nonzero semigroup right ideal of N and y is an element of N which centralizes U. Then, $y \in Z(N)$ by Lemma 4.2.13(ii).

Since y is an arbitrary element. Then every elements of N belongs to $Z(N)$.

Therefore, every element in N commutes under multiplication.

$i.e.$ $xy = yx$ for all $x, y \in N$

Hence, N is commutative under multiplication which also shows right distributive property.

Now we shall show additive commutativity of N.

Since U is nonzero semigroup ideal, then

$U^2 \neq \{0\}$ by Lemma 4.2.13(i)

So, there exist $z, w \in U$ $i.e.$ $zw \in U$ such that

$zw \neq 0$, then we get $zw \in U \subseteq Z(N)$

This implies that $zw \in Z(N)$.

Since $zw + zw = z(w + w) \in U \subseteq Z(N)$ for all $z \in U$, $w + w \in N$, then

$zw + zw \in Z(N)$.

Thus, $zw \neq 0$ and $zw \in Z(N)$ such that $zw + zw \in Z(N)$.

Then, by Lemma 4.1.17, we get

$(N, +)$ is Abelian.

Hence, N is a commutative ring.

Lemma 4.2.16. (Boua *et al.*, 2014) Let N be a prime near ring. If $N \subseteq Z(N)$, then N is a commutative ring.

Proof: Let $0 \neq z \in N \subseteq Z(N)$, this means $z \in Z(N)\setminus\{0\}$

Also $z + z \in N \subseteq Z(N)$, then $z + z \in Z(N)$ for all $z \in N$

For any $x, y \in N$

$$(z + z)(x + y) = (x + y)(z + z)$$

Now take $(z + z)(x + y) = (z + z)x + (z + z)y$ since N is left near ring

$= x(z + z) + y(z + z)$ since $z + z \in Z(N)$

$= xz + xz + yz + yz$

$= zx + zx + zy + zy$

$= z(x + x + y + y)$ (4.4)

Now again $(x + y)(z + z) = (x + y)z + (x + y)z$

$= z(x + y) + z(x + y)$ since $z \in Z(N)$

$= zx + zy + zx + zy$

$= z(x + y + x + y)$ (4.5)

From (4.4) and (4.5), we get

$$z(x + x + y + y) = z(x + y + x + y)$$

$$z(x + x + y + y) - z(x + y + x + y) = 0$$

$$z((x + x + y + y) - (x + y + x + y)) = 0$$

$((x + x + y + y) - (x + y + x + y)) = 0$ since $z \in Z(N)\setminus\{0\}$, then z is not a zero divisor

$$x + x + y + y = x + y + x + y$$

$$x + y = y + x \text{ for all } x, y \in N$$

Thus, $(N, +)$ is Abelian.

Since, $N \subseteq Z(N)$ then for any $x \in N$, $xy = yx$ for all $y \in N$. Thus, N is commutative.

Hence, N is a commutative ring.

5. SEMIGROUP IDEALS AND COMMUTATIVITY OF NEAR RINGS WITH SEMIDERIVATIONS

This section contains theorems concerning semigroup ideals and commutativity of near rings with semiderivations satisfying certain differential identities.
Further we discussed more details on the proofs of the theorems on semigroup ideals and commutativity of near rings satisfying certain differential identities involving semiderivations.

5.1. Commutativity of Near Rings with Semiderivations

The main purpose of this subsection elaborates the ideas and concepts of prime near rings satisfying certain identities involving semiderivations are commutative rings. There has been an ongoing interest concerning the relationship between the commutativity of a prime near ring N and the behavior of a semiderivation of N, with associated nonzero semiderivation. The more detailed on the proofs of the theorems on commutativity of near rings admitting a semiderivation satisfying certain differential identities have been provided.

The following theorems $(5.1.1 - 5.1.2)$ are the main results of semiderivation on near rings which facilitate commutativity of near rings.

Theorem 5.1.1 (Boua and Oukhtite, 2013) Let N be a 2-torsion free prime near ring. If N admits a nonzero semiderivation f such that $f(N) \subseteq Z(N)$, then N is a commutative ring.

Proof: We have $f(N) \subseteq Z(N)$, then
$f(xy) \in Z(N)$ for all $x, y \in N$
This implies that
$f(xy)g(z) = g(z)f(xy)$ for all $x, y, z \in N$
So, we have
$(xf(y) + f(x)g(y))g(z) = g(z)(xf(y) + f(x)g(y))$ for all $x, y, z \in N$
since $f(xy) = xf(y) + f(x)g(y)$
and by Lemma 4.2.7
$$xf(y)g(z) + f(x)g(yz) = g(z)xf(y) + g(z)f(x)g(y) \text{ for all } x, y, z \in N. \quad (5.1)$$

Replacing x by $f(x)$ in (5.1), we get

$$f(x)f(y)g(z) + f\big(f(x)\big)g(yz) = g(z)f(x)f(y) + g(z)f\big(f(x)\big)g(y) \text{ for all } x, y, z \in N$$

$$f(x)f(y)g(z) + f^2(x)g(yz) = g(z)f(x)f(y) + g(z)f^2(x)g(y) \text{ for all } x, y, z \in N$$

$$f(x)f(y)g(z) + f^2(x)g(yz) = f(x)f(y)g(z) + g(z)f^2(x)g(y) \text{ for all } x, y, z \in N$$

since $g(z)$ commute with $f(x)f(y)$

$$f^2(x)g(yz) = g(z)f^2(x)g(y)$$

$$f(f(x))g(yz) = g(z)f\big(f(x)\big)g(y)$$

Let $f(x) = r \in N$, then

$$f(r)g(yz) - g(z)f(r)g(y) = 0$$

$$f(r)g(yz) - f(r)g(z)g(y) = 0 \quad \text{since } f(r) \in Z(N)$$

$$f(r)\big(g(yz) - g(z)g(y)\big) = 0$$

$$f\big(f(x)\big)\big(g(yz) - g(z)g(y)\big) = 0$$

$$f^2(x)\big(g(yz) - g(z)g(y)\big) = 0$$

Replacing $g(yz) - g(z)g(y)$ by $t\big(g(yz) - g(z)g(y)\big)$ for all $t \in N$, then

$$f^2(x)t\big(g(yz) - g(z)g(y)\big) = 0 \text{ for all } x, y, z, t \in N$$

$$f^2(x)N(g(yz) - g(z)g(y)) = \{0\} \text{ for all } x, y, z \in N \qquad (5.2)$$

By the primeness of N, (5.2) assures that

$$g(yz) = g(z)g(y) \text{ or } f^2(x) = 0 \text{ for all } x, y, z \in N.$$

(i) If $g(yz) = g(z)g(y)$ for all $y, z \in N$, then (5.1), becomes

$$xf(y)g(z) + f(x)g(z)g(y) = g(z)xf(y) + g(z)f(x)g(y) \text{ for all } x, y, z \in N$$

$$xf(y)g(z) + f(x)g(z)g(y) = g(z)xf(y) + f(x)g(z)g(y) \text{ since } f(x) \in Z(N)$$

$$xf(y)g(z) = g(z)xf(y) \text{ for all } x, y, z \in N$$

$$xg(z)f(y) = g(z)xf(y) \quad \text{since } f(y) \in Z(N)$$

$$f(y)(xg(z)) = f(y)(g(z)x) \quad \text{since } f(y) \in Z(N)$$

$$f(y)(g(z)x) - f(y)(xg(z)) = 0$$

$$f(y)\big(g(z)x - xg(z)\big) = 0$$

$$f(y)[g(z), x] = 0$$

Replacing x by tx, then

$$f(y)[g(z), tx] = 0 \quad \text{for all } t \in N$$

$$f(y)t[g(z), x] = 0 \quad \text{for all } x, y, z, t \in N$$

$$f(y)N[g(z),x] = \{0\} \text{ for all } x,y,z \in N \tag{5.3}$$

Again by the primeness of N, equation (5.3), becomes

$$g(z) \in Z(N) \text{ for all } z \in N \tag{5.4}$$

We conclude that g is a homomorphism. Since $g(z)$ commute with $g(y)$, then

$g(yz) = g(y)g(z)$

Let $x,y,z \in N$, then we have

$$\begin{aligned} f(x(yz)) &= g(x)f(yz) + f(x)yz \\ &= g(x)(g(y)f(z) + f(y)z) + f(x)yz \\ &= g(x)g(y)f(z) + g(x)f(y)z + f(x)yz \\ &= g(xy)f(z) + g(x)f(y)z + f(x)yz \qquad (5.5) \end{aligned}$$

since g is homomorphism, then $g(x)g(y) = g(xy)$

On the other hand,

$$\begin{aligned} f((xy)z) &= g(xy)f(z) + f(xy)z \\ &= g(xy)f(z) + (g(x)f(y) + f(x)y)z \qquad (5.6) \end{aligned}$$

Comparing (5.5) with (5.6), we find that

$$(g(x)f(y) + f(x)y)z = g(x)f(y)z + f(x)yz \text{ for all } x,y,z \in N$$

since $f(xy) = g(x)f(y) + f(x)y \in Z(N)$ for all $x,y,z \in N$, it follows

$$(g(x)f(y) + f(x)y)z = z(g(x)f(y) + f(x)y) \text{ for all } x,y,z \in N$$

Hence by Lemma 4.2.7

$$g(x)f(y)z + f(x)yz = zg(x)f(y) + zf(x)y \text{ for all } x,y,z \in N$$

Using $f(N) \subseteq Z(N)$ together with (5.4), we get

$$g(x)f(y)z + f(x)yz = g(x)zf(y) + zf(x)y$$

$$g(x)f(y)z + f(x)yz = g(x)f(y)z + zf(x)y$$

$$f(x)yz = zf(x)y$$

$$f(x)yz = f(x)zy \quad \text{since } z \text{ commutes with } f(x)$$

$$f(x)[y,z] = 0 \text{ for all } x,y,z \in N$$

Replacing y by ty for all $t \in N$, then we get

$$f(x)[ty,z] = 0$$

$$f(x)t[y,z] = 0 \quad \text{for all } x,y,z,t \in N$$

$$f(x)N[y,z] = \{0\} \text{ for all } x,y,z \in N$$

By the primeness of N, $f(x) = 0$ or $[y,z] = 0$

Since f cannot be zero, then we have $[y,z] = 0$

So that $N \subseteq Z(N)$. Since every element of N commutes.

Hence, by applying Lemma 4.2.16, we conclude that N is a commutative ring.

(ii) If $f^2(x) = 0$ for all $x \in N$, then

$$\begin{aligned}
0 &= f^2(xy) \\
&= f\big(f(xy)\big) \\
&= f(g(x)f(y) + f(x)y) \\
&= f\big(g(x)f(y)\big) + f(f(x)y) \quad \text{since } f \text{ is additive} \\
&= g\big(g(x)\big)f\big(f(y)\big) + f\big(g(x)\big)f(y) + g\big(f(x)\big)f(y) + f\big(f(x)\big)y \\
&= g^2(x)f^2(y) + f(g(x))f(y) + g(f(x))f(y) + f^2(x)y \\
&= 2f(g(x))f(y) \qquad \text{since } g\big(f(x)\big) = f\big(g(x)\big)
\end{aligned}$$

Since N is 2-torsion free, it follows $f(g(x))f(y) = 0$ for all $x, y \in N$.

Hence applying Lemma 4.2.5 we get $f(g(x)) = 0$ for all $x \in N$.

Moreover, from

$$f(xg(y)) = g(x)f(g(y)) + f(x)g(y) \in Z(N) \text{ for all } x, y \in N$$

it follows

$$f(x)g(y) \in Z(N) \text{ for all } x, y \in N$$

According to Lemma 4.1.18, then we get

$$g(y) \in Z(N) \text{ for all } y \in N$$

Hence Equation (5.1) reduces to

$xf(y)g(z) + f(x)g(yz) = g(z)xf(y) + g(z)f(x)g(y)$ for all $x, y, z \in N$

$g(z)xf(y) + f(x)g(yz) = g(z)xf(y) + g(z)f(x)g(y)$ since $g(z)$ commutes

$f(x)g(yz) = g(z)f(x)g(y)$ for all $x, y, z \in N$.

$g(yz)f(x) = g(z)g(y)f(x)$ for all $x, y, z \in N$. Since $f(x)$ commutes with g

$g(yz)f(x) - g(z)g(y)f(x) = 0$

$\big(g(yz) - g(z)g(y)\big)f(x) = 0$ for all $x, y, z \in N$

In view of Lemma 4.2.5, we obtain

$$\begin{aligned}
g(yz) &= g(z)g(y) \\
&= g(y)g(z) \text{ for all } y, z \in N.
\end{aligned}$$

Since the last equation is the same as (i), then using the same argument we conclude that N is a commutative ring.

Theorem 5.1.2 (Boua and Oukhtite, 2013) Let N be a 2-torsion free prime near ring. If N admits a nonzero semiderivation f such that $f([x,y]) = 0$ for all $x, y \in N$, then N is a commutative ring.

Proof: Assume that

$$f([x,y]) = 0 \text{ for all } x, y \in N \tag{5.7}$$

Replacing y by xy in (5.7), because of $[x,xy] = x[x,y]$, we get

$$\begin{aligned} 0 &= f(x[x,y]) \\ &= g(x)f([x,y]) + f(x)[x,y] \quad \text{using semiderivation formula} \\ &= f(x)[x,y] \text{ for all } x, y \in N \end{aligned}$$

$$f(x)yx = f(x)xy \text{ for all } x, y \in N \tag{5.8}$$

Substituting yz for y in (5.8) and using (5.8), we obtain

$$\begin{aligned} f(x)yzx &= f(x)xyz \\ &= f(x)yxz \quad \text{for all } x, y, z \in N \end{aligned}$$

$$f(x)yxz - f(x)yzx = 0$$

$$f(x)y(xz - zx) = 0$$

$$f(x)y[x,z] = 0 \text{ for all } x, y, z \in N$$

$$f(x)N[x,z] = \{0\} \text{ for all } x, z \in N \tag{5.9}$$

By the primeness of N, equation (5.9) assures that

$$f(x) = 0 \text{ or } x \in Z(N) \text{ for all } x \in N \tag{5.10}$$

If there is an element $z \in N$ such that $f(z) = 0$, then we have both of

$$\begin{aligned} f\big((xy)z\big) &= f(xy)z + g(xy)f(z) \\ &= f(xy)z \\ &= (f(x)y + g(x)f(y))z \end{aligned}$$

and

$$\begin{aligned} f(x(yz)) &= f(x)yz + g(x)f(yz) \\ &= f(x)yz + g(x)(f(y)z + g(y)f(z)) \\ &= f(x)yz + g(x)f(y)z \end{aligned}$$

Consequently,

$$(f(x)y + g(x)f(y))z = f(x)yz + g(x)f(y)z \text{ for all } x, y, z \in N \qquad (5.11)$$

Since $f(xy) = f(yx)$, then

$$xf(y) + f(x)g(y) = g(y)f(x) + f(y)x \text{ for all } x, y \in N$$

Taking z instead of x, then we get

$$zf(y) + f(z)g(y) = g(y)f(z) + f(y)z$$

$$zf(y) = f(y)z \text{ for all } y, z \in N \qquad (5.12)$$

Replacing y by $f(x)y$ in (5.12) and using (5.11), we obtain

$$zf(f(x)y) = f(f(x)y)z$$

$$z\Big(f\big(f(x)\big)y + g\big(f(x)\big)f(y)\Big) = \Big(f\big(f(x)\big)y + g\big(f(x)\big)f(y)\Big)z$$

$$zf^2(x)y + zg(f(x))f(y) = f^2(x)yz + g(f(x))f(y)z \text{ for all } x, y, z \in N$$

since $g(f(x)) = f(g(x))$, then application of (5.12) implies that

(since by equation (5.12) $zf(y) = f(y)z$ implies z and $f(y)$ commutes)

$$zf^2(x)y + zf(g(x))f(y) = f^2(x)yz + f(g(x))f(y)z$$

$$zf^2(x)y + f(g(x))zf(y) = f^2(x)yz + f(g(x))f(y)z$$

$$zf^2(x)y + f(g(x))f(y)z = f^2(x)yz + f(g(x))f(y)z$$

$$zf^2(x)y = f^2(x)yz \text{ for all } x, y, z \in N \qquad (5.13)$$

Replacing y by yt in (5.13), we get

$$f^2(x)ytz = zf^2(x)yt$$

$$= f^2(x)yzt \text{ for all } x, y, z, t \in N$$

$$f^2(x)yzt - f^2(x)ytz = 0$$

$$f^2(x)y[z, t] = 0 \text{ for all } x, y, z, t \in N$$

$$f^2(x)N[z, t] = \{0\}$$

By the primeness of N, we get

$$f^2(x) = 0 \text{ or } z \in Z(N)$$

Therefore, equation (5.10) becomes

$$f^2(x) = 0 \text{ for all } x \in N \text{ or } N \subseteq Z(N)$$

If $f^2(x) = 0$ for all $x \in N$, then

$$0 = f^2(xy) \text{ for all } y \in N, xy \in N$$

$$= f\big(f(xy)\big)$$

$$= f(g(x)f(y) + f(x)y)$$
$$= f\big(g(x)f(y)\big) + f(f(x)y)$$
$$= g\big(g(x)\big)f\big(f(y)\big) + f\big(g(x)\big)f(y) + g\big(f(x)\big)f(y) + f\big(f(x)\big)y$$
$$= g^2(x)f^2(y) + 2f(g(x))f(y) + f^2(x)y$$
$$= 2f(g(x))f(y) \quad \text{since } g\big(f(x)\big) = f\big(g(x)\big)$$

so that by 2-torsion free $f(g(x))f(y) = 0$ for all $x, y \in N$.

By Lemma 4.2.5, it follows that

$$f(g(x)) = g(f(x))$$
$$= 0 \text{ for all } x \in N$$

Using the fact that f is a semiderivation, we get

$$f\big(xf(y)\big) = xf\big(f(y)\big) + f(x)g\big(f(y)\big)$$
$$= xf^2(y) + f(x)g(f(y))$$
$$f\big(xf(y)\big) = g(x)f\big(f(y)\big) + f(x)f(y)$$
$$= g(x)f^2(y) + f(x)f(y) \text{ for all } x, y \in N$$

which implies that

$$f(x)f(y) = 0 \quad \text{for all } x, y \in N.$$

Thus, Lemma 4.2.5 assures that $f(x) = 0$ for all $x \in N$; a contradiction.

Consequently, $N \subseteq Z(N)$.

Hence, by applying Lemma 4.2.16, we conclude that N is a commutative ring.

The following theorems $(5.1.3 - 5.1.7)$ show that semiderivation on near rings satisfying certain differential identities which facilitate near rings are commutative rings. In this subsection g is considered to be as a surjective function.

Theorem 5.1.3 (Kim, 2016) Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x\ o\ y) = x\ o\ y$ for all $x, y \in N$, then N is commutative ring.

Proof: By hypothesis, we have

$$f(x\ o\ y) = x\ o\ y \text{ for all } x, y \in N \tag{5.14}$$

Replacing y by xy in (5.14), we get

$$f(x\ o\ (xy)) = xo(xy) \quad \text{for all } x, y \in N \tag{5.15}$$

$$f(x(xoy)) = x(xoy) \qquad \text{since } xo(xy) = x(xoy)$$

$$f(x)g(x\ o\ y) + xf(x\ o\ y) = x(xoy) \text{ for all } x, y \in N$$

using equation (5.14), we get

$f(x)(x\ o\ y) + x(x\ o\ y) = x(xoy)$ for all $x, y \in N$ since g surjective, then $g(x\ o\ y) = xoy$

$$f(x)(x\ o\ y) = 0 \text{ for all } x, y \in N$$

$$f(x)(xy + yx) = 0$$

$$f(x)xy = -f(x)yx \text{ for all } x, y \in N \tag{5.16}$$

Substituting yz for y in (5.16), we obtain for all $x, y \in N$

$$-f(x)yzx = f(x)xyz = (-f(x)yx)z = f(x)y(-x)z \tag{5.17}$$

Since $-f(x)yzx = f(x)y(-x)z$, (5.17) becomes

$$f(x)yz(-x) = f(x)y(-x)z \text{ for all } x, y, z \in N \tag{5.18}$$

Taking $-x$ instead of x in (5.18), we obtain

$$f(-x)yzx = f(-x)yxz \text{ for all } x, y, z \in N \tag{5.19}$$

$$f(-x)yzx - f(-x)yxz = 0$$

$$f(-x)y(zx - xz) = 0 \text{ for all } x, y, z \in N$$

$$f(-x)N[z, x] = \{0\} \text{ for all } x, z \in N \tag{5.20}$$

By primeness, we have either $x \in Z(N)$ or $f(-x) = 0$

That is,

$$f(x) = 0 \text{ or } [x, z] = 0 \text{ for all } x, z \in N \tag{5.21}$$

From (5.21), it follows that for each fixed $x \in N$, we have

$$f(x) = 0 \text{ or } x \in Z(N) \tag{5.22}$$

But $x \in Z(N)$ also implies that $f(x) \in Z(N)$, and

so from equation (5.20) , we get

$$f(x) \in Z(N) \text{ for all } x \in N \tag{5.23}$$

from equation (5.23), we have

$f(N) \subseteq Z(N)$ and using theorem 5.1.1, we conclude that N is commutative ring.

Theorem 5.1.4 (Kim, 2016) Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x)oy = xoy$ for all $x, y \in N$, then N is commutative ring.

Proof. Suppose that

$$f(x)\ o\ y\ =\ x\ o\ y \text{ for all } x,y \in\ N \tag{5.24}$$

Replacing x by xy in (5.24), we get

$$f(xy)\ o\ y\ =\ xy\ o\ y\ =\ (x\ o\ y)y$$

$$f(xy)\ o\ y\ =\ (x\ o\ y)y$$

$$f(xy)\ o\ y\ =\ (f(x)oy)y \quad \text{since } x\ o\ y = f(x)\ o\ y \text{ equation 5.24}$$

$$f(xy)y\ +\ yf(xy) = \big(f(x)y + yf(x)\big)y$$

$$f(xy)y\ +\ yf(xy)\ =\ f(x)y^2\ +\ yf(x)y$$

$$\big(f(x)g(y) + xf(y)\big)y + y\big(f(x)g(y) + xf(y)\big) = f(x)y^2 + yf(x)y$$

$$f(x)g(y)y + xf(y)y + yf(x)g(y) + yxf(y) = f(x)y^2 + yf(x)y$$

Hence we have

$$f(x)y^2\ +\ xf(y)y\ +\ yf(x)y\ +\ yxf(y)\ =\ f(x)y^2\ +\ yf(x)y$$

since g is surjective $g(y) = y$.

Therefore, we obtain

$$xf(y)y + yxf(y)\ =\ 0 \text{ for all } x,y \in\ N$$

$$yxf(y) =\ -xf(y)y \quad \text{for all } x,y \in\ N \tag{5.25}$$

Replacing x by xz in equation (5.25), we get for all $x,y \in\ N$,

$$yxzf(y) =\ -xzf(y)y\ =\ -x(zf(y)y)$$

$$=\ -x\big(-yzf(y)\big) =\ -x(-y)zf(y)$$

The last expression reduced to

$$yxzf(y) =\ -x(-y)zf(y) \text{ for all } x,y,z \in\ N \tag{5.26}$$

$$(-y)xzf(y)\ =\ x(-y)zf(y) \text{ for all } x,y,z \in\ N \tag{5.27}$$

Taking $-y$ instead of y in (5.27), we get

$$yxzf(-y)\ =\ xyzf(-y) \text{ for all } x,y,z \in\ N$$

$$yxzf(-y)\ -\ xyzf(-y) = 0$$

$$(yx -\ xy)zf(-y)\ =\ 0$$

$$[y,x]zf(-y)\ =\ 0 \quad \text{for all } x,y,z \in\ N$$

$$[y,x]Nf(-y)\ =\ \{0\} \quad \text{for all } x,y \in\ N \tag{5.28}$$

By primeness, we have either $y \in\ Z(N)$ or $f(-y)\ =\ 0$ for all $y \in\ N$

$$f(y)\ =\ 0 \text{ or } y \in\ Z(N) \text{ for all } y \in\ N \tag{5.29}$$

But $y \in Z(N)$ also implies that $f(y) \in Z(N)$, and

so equation (5.28) becomes

$$f(y) \in Z(N) \text{ for all } y \in N \tag{5.30}$$

From equation (5.30), we get

$f(N) \subseteq Z(N)$ and using theorem 5.1.1, we conclude that N is commutative ring.

Theorem 5.1.5 (Kim, 2016) Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f(x\, o\, y) = [x, y]$ for all $x, y \in N$, then N is commutative ring.

Proof. Let

$$f(x\, o\, y) = [x, y] \text{ for all } x, y \in N \tag{5.31}$$

Replacing y by yx in (5.31), we obtain

$$f\big(xo(yx)\big) = [x, yx]$$

$$f\big((xoy)x\big) = [x, y]x$$

$$f(x\, o\, y)g(x) + (x\, o\, y)f(x) = [x, y]x$$

$$[x, y]x + (xoy)f(x) = [x, y]x \quad \text{since } g \text{ is onto}$$

$$(x\, o\, y)f(x) = 0 \text{ for all } x, y \in N \tag{5.32}$$

Again replacing y by zy in (5.32), we obtain

$$\big(xo(zy)\big)f(x) = 0$$

$$(x(zy) + (zy)x)f(x) = 0 \text{ for all } x, y, z \in N$$

Now, from application of (5.32), we have

$$(xoy)f(x) = 0$$

$$(xy + yx)f(x) = 0$$

$$(xy)f(x) + (yx)f(x) = 0$$

$$yxf(x) = -xyf(x)$$

Combining this fact with the latter relation, we have

$$(x(zy) + (zy)x)f(x) = 0$$

$$x(zy)f(x) + (zy)xf(x) = 0$$

$$xzyf(x) + zyxf(x) = 0$$

$$(xz)yf(x) + z\big(-xyf(x)\big) = 0 \quad \text{since } yxf(x) = -xyf(x)$$

$$(xz)yf(x) + z(-x)yf(x) = 0$$

$$\left(xz + z(-x)\right)yf(x) = 0$$

$$(xz - zx)yf(x) = 0$$

$$[x,z]yf(x) = 0 \quad \text{for all } x,y,z \in N$$

$$[x,z]Nf(x) = \{0\} \quad \text{for all } x,z \in N \tag{5.33}$$

Since N is a prime near-ring, for each $x \in N$, we have either

$$f(x) = 0 \text{ or } x \in Z(N) \text{ for all } x \in N$$

But $x \in Z(N)$ also implies that $f(x) \in Z(N)$,

so equation (5.33) becomes

$$f(x) \in Z(N) \quad \text{for all } x \in N \tag{5.34}$$

From equation (5.34), we get

$f(N) \subseteq Z(N)$ and using theorem 5.1.1, we conclude that N is commutative ring.

Theorem 5.1.6 (Kim, 2016) Let N be a prime near-ring. If N admits a nonzero semiderivation f with g such that $f[x,y] = (x\ o\ y)$ for all $x,y \in N$, then N is commutative ring.

Proof. Let

$$f([x,y]) = (x\ o\ y) \quad \text{for all } x,y \in N \tag{5.35}$$

Replacing y by yx in (5.35), we get for all $x,y \in N$

$$f([x,yx]) = (xoyx)$$

$$f([x,y]x) = (x\ o\ y)x$$

Using semiderivation formula, we obtain

$$f([x,y])g(x) + [x,y]f(x) = (x\ o\ y)x$$

$$(x\ o\ y)x + [x,y]f(x) = (x\ o\ y)x$$

Since g is onto and using equation (5.35), we get

$$[x,y]f(x) = 0 \tag{5.36}$$

Replacing y by yz in (5.36), we obtain

$$[x,yz]f(x) = 0$$

$$[x,y]zf(x) = 0 \quad \text{for all } x,y,z \in N$$

$$[x,y]Nf(x) = \{0\} \quad \text{for all } x,y \in N \tag{5.37}$$

Since N is a prime near-ring, for each $x \in N$, we have either

$$f(x) = 0 \text{ or } x \in Z(N) \text{ for all } x \in N$$

But $x \in Z(N)$ also implies that $f(x) \in Z(N)$,

so that equation (5.37) becomes

$$f(x) \in Z(N) \text{ for all } x \in N \qquad (5.38)$$

From equation (5.38), we get

$f(N) \subseteq Z(N)$ and using theorem 5.1.1, we conclude that N is commutative ring.

Theorem 5.1.7 (Kim, 2016) Let N be a 2- torsion free prime near-ring and let f is a semiderivation associated with a surjective function $g: N \to N$. If $f^2(x) = 0$ for all $x \in N$, then $f = 0$.

Proof. By hypothesis, we have for all $x \in N$

$$f^2(x) = 0$$

Replacing x by xy in the above equation, we obtain

$$f^2(xy) = 0 \text{ for all } x, y \in N$$

Hence, for any $x, y \in N$,

$$\begin{aligned} 0 &= f(f(xy)) \\ &= f(f(x)g(y) + xf(y)) \\ &= f\big(f(x)g(y)\big) + f\big(xf(y)\big) \quad \text{since } f \text{ is additive} \\ &= f\big(f(x)\big)g\big(g(y)\big) + f(x)f\big(g(y)\big) + f(x)g\big(f(y)\big) + xf\big(f(y)\big) \\ &= f^2(x)g(g(y)) + f(x)f(g(y)) + f(x)g(f(y)) + xf^2(y) \\ &= 2f(x)f(g(y)) \qquad \text{since } g\big(f(y)\big) = f\big(g(y)\big) \end{aligned}$$

Since N is 2- torsion free and g surjective, we have

$$f(x)f(y) = 0 \text{ for all } x, y \in N$$

Replacing y by yz, we get for all $x, y, z \in N$,

$$\begin{aligned} 0 &= f(x)f(yz) \\ &= f(x)\big(f(y)g(z) + yf(z)\big) \\ &= f(x)f(y)g(z) + f(x)yf(z) \\ &= f(x)yf(z) \text{ for all } x, y, z \in N \quad \text{since } f(x)f(y) = 0 \\ &= f(x)Nf(z) \quad \text{for all } x, z \in N \end{aligned}$$

Since N is prime, we have $f(x) = 0$ or $f(z) = 0$ for all $x, z \in N$

Hence, in both cases, $f = 0$.

5.2. Semiderivation on Near Rings Satisfying Certain Differential Identities in the Setting of Semigroup Ideals

In this subsection the ideas and concepts on commutativity of a near ring involving semiderivation in the setting of semigroup ideals have been elaborated. Also, the more details proofs of the theorems on semigroup ideals and commutativity of near rings involving semiderivations satisfying certain differential identities has been provided. In this subsection a semiderivation f is associated with an automorphism g.

Theorem 5.2.1 (Boua *et al*., 2014) Let N be a prime near-ring and U be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation f, then the following assertions are equivalent:

(i) $f(U) \subseteq Z(N)$

(ii) N is a commutative ring.

Proof. (ii) ⇒ (i) since N is a commutative ring, then

$$[x, y] = 0 \text{ for all } x, y \in N$$

we replace x by $f(u)$

$$[f(u), y] = 0 \text{ for all } y \in N,\ u \in U$$

$$f(u)y = yf(u) \text{ for all } y \in N,\ u \in U$$

$f(u) \in Z(N)$ for all $u \in U$ since $f(u)$ commutes with all element of N.

Hence, $f(U) \subseteq Z(N)$

(i) ⇒ (ii) by the hypothesis given, we have

$$f(xy)z = zf(xy) \quad \text{for all } y \in U,\ x, z \in N \quad \text{since } f(U) \subseteq Z(N)$$

$$(f(x)y + g(x)f(y))z = z(f(x)y + g(x)f(y))$$

Taking Lemma 4.2.8(i), we get

$$f(x)yz + g(x)f(y)z = zf(x)y + zg(x)f(y) \text{ for all } y \in U,\ x, z \in N$$

Since $f(y)$ commute with every element of N, so that

$$f(x)yz + f(y)g(x)z = zf(x)y + f(y)zg(x) \text{ for all } y \in U,\ x, z \in N \qquad (5.39)$$

Substituting $g(x)$ for z in (5.39), we obtain

$$f(x)yg(x) + f(y)g(x)g(x) = g(x)f(x)y + f(y)g(x)g(x) \text{ for all } y \in U,\ x \in N$$

$$f(x)yg(x) = g(x)f(x)y \text{ for all } y \in U, x \in N \tag{5.40}$$

Replacing y by yt in (5.40) and using this, we get

$$f(x)ytg(x) = (g(x)f(x)y)t$$

$$= f(x)yg(x)t \quad \text{since } g(x)f(x)y = f(x)yg(x)$$

$$f(x)yg(x)t - f(x)ytg(x) = 0$$

$$f(x)y\big(g(x)t - tg(x)\big) = 0$$

$$f(x)y[g(x),t] = 0 \text{ for all } y \in U, x,t \in N$$

$$f(x)U[g(x),t] = \{0\} \text{ for all } x,t \in N$$

By Lemma 4.2.14, this implies that

$$f(x) = 0 \text{ or } g(x) \in Z(N) \text{ for all } x \in N \tag{5.41}$$

Taking the fact that $f \neq 0$, then (5.41) shows that

there is an element $x_o \in N$ such as

$$g(x_o) \in Z(N) \text{ and } f(x_o) \neq 0$$

In this case, equation (5.39) becomes

$$f(x_o)yz = zf(x_o)y \text{ for all } y \in U, z \in N$$

Again replacing y by yt, we get

$$f(x_o)ytz = (zf(x_o)y)t \quad \text{for all } y \in U, z,t \in N$$

$$= f(x_o)yzt \text{ since } zf(x_o)y = f(x_o)yz$$

$$f(x_o)yzt - f(x_o)ytz = 0$$

$$f(x_o)y(zt - tz) = 0$$

$$f(x_o)y[z,t] = 0 \text{ for all } y \in U, z,t \in N$$

$$f(x_o)U[z,t] = \{0\} \text{ for all } z,t \in N \tag{5.42}$$

Taking Lemma 4.2.14, (5.42) implies that

$$f(x_o) = 0 \text{ or } N \subseteq Z(N)$$

Since the first of these conditions is impossible, the second must hold.

Hence, N is commutative ring by Lemma 4.2.16.

Theorem 5.2.2 (Boua *et al.*, 2014) Let N be a 2-torsion free prime near-ring and U be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation f, then the following assertions are equivalent:

(i) $f(-U) \subseteq Z(N)$

(ii) N is a commutative ring.

Proof. For (ii) ⇒ (i) since N is a commutative ring, then

$$[x,y]=0 \text{ for all } x,y\in N$$

we replace x by $f(-u)$

$$[f(-u),y]=0 \text{ for all } y\in N, u\in U$$

$$f(-u)y=yf(-u) \text{ for all } y\in N$$

$f(-u)\in Z(N)$ for all $u\in U$ since $f(-u)$ commutes with all element of N

Hence, $f(-U)\subseteq Z(N)$

(i) ⇒ (ii) We have $f(-x)\in Z(N)$ for all $x\in U$, then

$$f(-tx) = f(t(-x)) \in Z(N) \text{ for all } x\in U,\ t\in N \tag{5.43}$$

In particular, for all $t\in Z(N)$ we have

$$f(t(-x)) = tf(-x) + f(t)g(-x) \in Z(N) \text{ for all } x\in U$$

$$f(t)g(-x)\in Z(N) \text{ for all } x\in U \tag{5.44}$$

Since g is an automorphism, then $f(t)\in Z(N)$

By the application of Lemma 4.1.18, (5.44) yields

$$f(t) = 0 \text{ or } g(-x)\in Z(N) \text{ for all } x\in U,\ t\in Z(N) \tag{5.45}$$

If $f(Z(N)) = \{0\}$, taking (5.43) into account, we get

$$f\Big(f\big(t(-x)\big)\Big)=0 \text{ for all } x\in U,\ t\in N$$

$$f^2(t)(-x) + 2g(f(t))f(-x) = 0 \text{ for all } x\in U,\ t\in N \tag{5.46}$$

Replacing t by $f(t)$ in (5.46), we get

$$f^3(t)(-x) + 2g(f^2(t))f(-x) = 0 \text{ for all } x\in U,\ t\in N \tag{5.47}$$

On the other hand, applying f for (5.46), we find that

$$f^3(t)(-x) + 3g(f^2(t))f(-x) = 0 \text{ for all } x\in U,\ t\in N \tag{5.48}$$

From (5.47) and (5.48), we conclude that $g(f^2(t))f(-x) = 0$ for all $x\in U,\ t\in N$

Taking the fact that $f(-x)\in Z(N)$, then

$$f^2\big(g(t)\big)Nf(-x) = \{0\} \text{ for all } x\in U,\ t\in N$$

By the primeness of N, the last equation implies that

$$f^2=0 \text{ or } f=0 \tag{5.49}$$

If $f^2=0$, then $f=0$ by theorem 5.1.2.

Therefore (5.49) shows that $f = 0$, a contradiction. Since $f \neq 0$ by hypothesis.

Consequently $f(Z(N)) \neq \{0\}$ and (5.45) prove that $g(-x) \in Z(N)$ for all $x \in U$.

Let $v \in N$ and $x \in U$, we have $g(-vx) = g(v)g(-x) \in Z(N)$,

by Lemma 4.1.18, we get

$$g(-x) = 0 \text{ or } g(v) \in Z(N) \text{ for all } x \in U, \ v \in N \tag{5.50}$$

(i) If $g(-x) = 0$ for all $x \in U$, by this hypothesis we have

$$f(-yx) = f(y)g(-x) + yf(-x) \in Z(N) \text{ for all } x \in U, \ y \in N$$

$$yf(-x) \in Z(N) \text{ for all } x \in U, \ y \in N$$

Using Lemma 4.1.18 and taking the fact $f \neq 0$, we arrive at $N \subseteq Z(N)$

Applying Lemma 4.2.16, we conclude that N is commutative ring.

(ii) If there is an element $x_o \in U$ such that $g(-x_o) \neq 0$, then equation (5.50) shows that

$$g(v) \in Z(N) \text{ for all } v \in N$$

Since g is an automorphism we conclude that $N \subseteq Z(N)$. Thus N is commutative ring.

Theorem 5.2.3 (Boua *et al*., 2014) Let N be a prime near-ring and U be a nonzero semigroup ideal of N. If N admits a semiderivation f, then the following assertions are equivalent:

(i) $f([x,y]) = 0$ for all $x, y \in U$.

(ii) $f([x,y]) = [x,y]$ for all $x, y \in U$.

(iii) N is a commutative ring.

Proof. Since N is commutative ring

For all $u, v \in N$, then $uv = vu$

Since U is a subset of N then it is also true for U.

Thus, $xy = yx$ for all $x, y \in U$, U is commutative under multiplication.

(iii) ⇒ (i) Since N is commutative ring, then

$$xy = yx \text{ for all } x, y \in U$$

$$f(xy) = f(yx)$$

$$f(xy) - f(yx) = 0$$

$$f(xy - yx) = 0 \text{ as } f \text{ is additive}$$

$$f([x,y]) = 0 \text{ for all } x, y \in U.$$

(iii) ⇒ (ii) Since N is commutative ring, then

$$xy = yx \text{ for all } x, y \in U$$

We need to show that $f([x,y]) = [x,y]$ for all $x, y \in U$

To see left hand side $f([x,y]) = f(xy - yx) = f(0) = 0$ since $xy = yx$ for all $x, y \in U$

To see right hand side $[x,y] = xy - yx = 0$ since $xy = yx$ for all $x, y \in U$

Thus, $f([x,y]) = [x,y]$

Proving that (i) ⇒ (iii) Suppose that

$$f([x,y]) = 0 \text{ for all } x, y \in U \tag{5.51}$$

Substituting xy for y in (5.51), we have

$$f([x,xy]) = 0$$

$$f(x[x,y]) = 0 \quad \text{since } [x,xy] = x[x,y]$$

$$f(x)[x,y] + g(x)f[x,y] = 0 \text{ for all } x, y \in U$$

$$f(x)[x,y] = 0$$

$$f(x)(xy - yx) = 0$$

$$f(x)xy = f(x)yx \text{ for all } x, y \in U \tag{5.52}$$

Replacing y by yt in (5.52) and using this, we get

$$f(x)xyt = f(x)ytx$$

$$f(x)yxt = f(x)ytx \quad \text{since } f(x)xy = f(x)yx$$

$$f(x)yxt - f(x)ytx = 0$$

$$f(x)y(xt - tx) = 0$$

$$f(x)y[x,t] = 0 \quad \text{for all } x, y \in U, t \in N$$

$$f(x)U[x,t] = \{0\} \quad \text{for all } x \in U, t \in N$$

Taking into account the Lemma 4.2.14, we get

$$f(x) = 0 \text{ or } x \in Z(N) \text{ for all } x \in U \tag{5.53}$$

Since f is associated with an automorphism, we have $f(x) \in Z(N)$ for each $x \in Z(N)$, then (5.53) illustrated $f(U) \subseteq Z(N)$.

Hence, N is commutative ring by the use of Theorem 5.2.1.

Proving that (ii) ⇒ (iii) By the hypothesis given, we have

$$f([x,y]) = [x,y] \text{ for all } x, y \in U \tag{5.54}$$

Replacing y by xy in (5.54), we get

$$f([x,xy]) = [x,xy]$$

$$f(x[x,y]) = x[x,y]$$

$$xf([x,y]) + f(x)g([x,y]) = x[x,y] \text{ for all } x,y \in U$$

$$x[x,y] + f(x)g([x,y]) = x[x,y]$$

$$f(x)g([x,y]) = 0$$

$$f(x)g(xy - yx) = 0$$

$$f(x)g(xy) - f(x)g(yx) = 0$$

$$f(x)g(xy) = f(x)g(yx)$$

$$f(x)g(x)g(y) = f(x)g(y)g(x) \text{ since } g \text{ is an automorphism}$$

$$f(x)g(x)g(y) = f(x)g(y)g(x) \text{ for all } x,y \in U \quad (5.55)$$

Since g is an automorphism, (5.55) shows that

$$f(x)g(x)j = f(x)jg(x) \text{ for all } x \in U,\ j \in J \quad (5.56)$$

with $J = g(U)$, it is clear that J is a semigroup ideal of N

Substituting jz for j in (5.56) and using this, we obtain

$$f(x)g(x)jz = f(x)jzg(x)$$

$$f(x)jg(x)z = f(x)jzg(x) \text{ since } f(x)g(x)j = f(x)jg(x)$$

$$f(x)jg(x)z - f(x)jzg(x) = 0$$

$$f(x)j\big(g(x)z - zg(x)\big) = 0$$

$$f(x)j[g(x),z] = 0 \text{ for all } x \in U,\ j \in J,\ z \in N \quad (5.57)$$

$$f(x)J[g(x),z] = \{0\} \text{ for all } x \in U,\ z \in N \quad (5.58)$$

By the application of Lemma 4.2.14, (5.58) becomes

$$f(x) = 0 \text{ or } g(x) \in Z(N) \text{ for all } x \in U$$

$$f(g(x)) \in Z(N) \text{ for all } x \in U \quad \text{since } f \text{ is an automorphism, then } f\big(g(x)\big) = g(x)$$

Consequently, we deduce that $f(J) \subseteq Z(N)$

Hence, N is a commutative ring by Theorem 5.2.1.

Theorem 5.2.4 (Boua *et al*., 2014) Let N be a 2-torsion free prime near-ring and U be a semigroup ideal of N, then N admits no nonzero semiderivation f satisfying one of the assertions as the following.

(i) $f(x\ o\ y) = 0$ for all $x,y \in U$.

(ii) $f(x\ o\ y) = (x\ o\ y)$ for all $x,y \in U$.

Proof. (i) Suppose that f is nonzero semiderivation which satisfy the following

$$f(x\ o\ y)\ =\ 0\ \text{ for all } x,y\in U \tag{5.59}$$

Replacing y by xy in (5.59) and taking the fact that $x\ o\ xy\ =\ x(x\ o\ y)$, we get

$$f(x\ o\ (xy))\ =\ 0$$

$$f(x(x\ o\ y))\ =\ 0$$

$$f(x)(x\ o\ y)+g(x)f(x\ o\ y)\ =\ 0$$

$$f(x)(x\ o\ y)\ =\ 0\ \text{ for all } x,y\in U$$

$$f(x)(xy+yx)=0$$

$$f(x)xy\ =\ -f(x)yx\ \text{ for all } x,y\in U \tag{5.60}$$

Substituting yt for y in (5.60), we obtain

$$f(x)xyt\ =\ -f(x)ytx$$

$$-f(x)yxt\ =\ -f(x)ytx\quad \text{since }\ f(x)xy\ =\ -f(x)yx$$

$$f(x)y(-x)t\ =\ f(x)yt(-x)\ \text{ for all } x,y\in U,t\in N$$

$$f(x)y(-x)t-f(x)yt(-x)=0$$

$$f(x)y[-x,t]=0$$

$$f(x)U[-x,t]\ =\ \{0\}\ \text{ for all } x\in U,t\in N$$

By using lemma 4.2.14, we have

$$f(x)\ =\ 0\ \text{ or }\ -x\ \in\ Z(N)\ \text{ for all } x\in U$$

Thus, $f(-x)\ \in\ Z(N)$ for all $x\in U$, it means that

$$f(-U)\subseteq Z(N)$$

Hence, N is a commutative ring according to theorem 5.2.2.

In this case, returning to the hypothesis given, we have

$$f(x\ o\ y)=0$$

$$f(xy+yx)=0$$

$$f(2xy)=0\qquad \text{since } xy=yx$$

$$2f(xy)=0\qquad \text{since } N \text{ is 2-torsion free}$$

$$f(xy)\ =\ 0\quad \text{for all } x,y\in U$$

$$f(x)y\ +\ g(x)f(y)\ =\ 0\quad \text{for all } x,y\in U \tag{5.61}$$

Taking yz instead of y in (5.61), we get

$$f(x)yz+g(x)f(yz)\ =\ 0$$

$$f(x)yz\ =\ 0\ \text{ for all } x,y,z\in U\quad \text{since } f(yz)=0$$

$$f(x)Uz = \{0\} \text{ for all } x, z \in U$$

By Lemma 4.2.14, the last expression shows that $f = 0$, which is a contradiction to our assumption.

Thus, condition (i) is true only when f is zero semiderivation.

(ii) Suppose that f is nonzero semiderivation such that

$$f(x\ o\ y) = x\ o\ y \text{ for all } x, y \in U \tag{5.62}$$

Putting xy instead of y in (5.62), we get

$$f(x\ o\ xy) = x\ o\ xy$$

$$f(x(x\ o\ y)) = x(x\ o\ y) \quad \text{since } x\ o\ xy = x(x\ o\ y)$$

$$xf(x\ o\ y) + f(x)g(x\ o\ y) = x(x\ o\ y)$$

$$x(x\ o\ y) + f(x)g(x\ o\ y) = x(x\ o\ y)$$

$$f(x)g(x\ o\ y) = 0$$

$$f(x)g(xy + yx) = 0$$

$$f(x)g(xy) = -f(x)g(yx)$$

$$f(x)g(x)g(y) = -f(x)g(y)g(x) \text{ for all } x, y \in U \text{ since } g \text{ is an automorphism}$$

Since g is an automorphism, we get

$$f(x)g(x)n = -f(x)ng(x) \text{ for all } x \in U, n \in J = g(U) \tag{5.63}$$

Writing nm instead of n in (5.63), we find that

$$f(x)g(x)nm = -f(x)nmg(x)$$

$$-f(x)ng(x)m = -f(x)nmg(x) \text{ since } f(x)g(x)n = -f(x)ng(x)$$

$$f(x)ng(-x)m - f(x)nmg(-x) = 0$$

$$f(x)n\big(g(-x)m - mg(-x)\big) = 0$$

$$f(x)n[g(-x), m] = 0 \text{ for all } x \in U, n \in J, m \in N$$

$$f(x)J[g(-x), m] = \{0\} \text{ for all } x \in U, m \in N \tag{5.64}$$

Applying Lemma 4.2.14, (5.64) shows that

$$f(x) = 0 \text{ or } g(-x) \in Z(N) \text{ for all } x \in U.$$

Therefore $f(g(-x)) \in Z(N)$ for all $x \in U$ since f is an automorphism, then $f\big(g(-x)\big) = g(-x)$

Consequently, we deduce that $f(-J) \subseteq Z(N)$

Hence, we conclude that N is a commutative ring by theorem 5.2.2.

In this case, returning to the hypothesis given, we have

$$f(x\ o\ y) = x\ o\ y$$
$$f(xy + yx) = xy + yx$$
$$f(2xy) = 2xy \quad \text{since } xy = yx$$
$$2f(xy) = 2xy$$
$$f(xy) = xy \quad \text{for all } x, y \in U$$
$$f(x)y + g(x)f(y) = xy \quad \text{for all } x, y \in U \tag{5.65}$$

Substituting xz for x in (5.65), we obtain

$$f(xz)y + g(xz)f(y) = xzy$$
$$xzy + g(xz)f(y) = xzy \qquad \text{since } f(xz) = xz$$
$$g(x)g(z)f(y) = 0 \quad \text{for all } x, y, z \in U \qquad \text{since } g \text{ is an automorphism}$$
$$g(x)Jf(y) = \{0\} \quad \text{for all } x, y \in U \tag{5.66}$$

By Lemma 4.2.14, (5.66) demonstrates that

$$g(U) = \{0\} \text{ or } f = 0$$

But each of these conditions yields a contradiction to our assumption.

Thus, condition (ii) hold only when f is zero semiderivation.

So, condition (i) and (ii) hold only for zero semiderivation.

Hence, N admits no nonzero semiderivation for which (i) and (ii) hold.

The following lemmas $(5.2.5 - 5.2.7)$ show that the extensions of some results of derivations on near rings to semiderivation on near rings which helps for the proofs of the next theorems.

Lemma 5.2.5 (Ali *et al*., 2016) Let N be a right near ring admitting a nonzero semiderivation f associated with a map g such that $g(xy) = g(x)g(y)$, for all $x, y \in N$. Then N satisfies the following partial distributive law:

$$x(f(y)g(z) + yf(z)) = xf(y)g(z) + xyf(z) \text{ for all } x, y, z \in N$$

Proof. We have $f(x(yz)) = f(x)g(yz) + xf(yz)$ for all $x, y, z \in N$

$$f(x(yz) = f(x)g(y)g(z) + x(f(y)g(z) + yf(z)) \tag{5.67}$$

On the other hand, $f((xy)z) = f(xy)g(z) + xyf(z)$ for all $x, y, z \in N$

$$f((xy)z) = (f(x)g(y) + xf(y))g(z) + xyf(z) \text{ for all } x, y, z \in N$$
$$f((xy)z) = f(x)g(y)g(z) + xf(y)g(z) + xyf(z) \tag{5.68}$$

From (5.67) and (5.68), we obtain

$$x(f(y)g(z) + yf(z)) = xf(y)g(z) + xyf(z) \text{ for all } x, y, z \in N.$$

Lemma 5.2.6 (Ali *et al*., 2016) Let N be a right prime near ring and U be a nonzero semigroup ideal of N. If f is a nonzero semiderivation of N associated with a map g, then $f \neq 0$ on U.

Proof. Let $f(u) = 0$ for all $u \in U$

Replacing u by ux, we get $f(ux) = 0$ for all $u \in U$ and $x \in N$

Thus $f(u)g(x) + uf(x) = 0$ for all $u \in U$ and $x \in N$

$$uf(x) = 0 \quad \text{since } f(u) = 0$$

$$f(x) = 0 \quad \text{by Lemma 4.2.13 } (i) \text{ which is a contradiction.}$$

Hence, for nonzero semiderivation f on N, $f \neq 0$ on U.

Lemma 5.2.7 (Ali *et al*., 2016) Let N be a right prime near ring and U be a nonzero semigroup ideal of N. Suppose f is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $a \in N$ and $af(U) = 0$ (or $f(U)a = 0$), then $a = 0$.

Proof. Let $af(u) = 0$ for all $u \in U$

Replacing u by uv, we get

$$af(uv) = 0 \text{ for all } u, v \in U$$

$$a(f(u)g(v) + uf(v)) = 0 \text{ for all } u, v \in U$$

Using Lemma 5.2.5, we get

$$af(u)g(v) + auf(v) = 0$$

$$auf(v) = 0 \text{ for all } u, v \in U \quad \text{since } af(u) = 0$$

$$aUf(v) = \{0\} \text{ for all } v \in U, a \in N$$

Choosing v such that $f(v) \neq 0$ and applying Lemma 4.2.14, we get $a = 0$.

The following theorems $(5.2.8 - 5.2.10)$ show that the extensions of some results of derivation on near rings to semiderivation on near rings in the setting of semigroup ideals which facilitate the commutativity of near rings.

Theorem 5.2.8 (Ali *et al*., 2016) Let N be a 2-torsion free zero-symmetric right prime near ring and U be a nonzero semigroup ideal of N. If f is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$, then $f^2(U) \neq \{0\}$.

Proof. Suppose $f^2(u) = 0$

$$f^2(uv) = 0 \text{ for all } u, v \in U$$

Now we exploit the definition of semiderivation f in different ways to obtain

$$\begin{aligned} 0 = f^2(uv) = f\big(f(uv)\big) &= f\big(f(u)v + g(u)f(v)\big) \\ &= f(f(u)v) + f\big(g(u)f(v)\big) \text{ since } f \text{ is additive} \\ &= f\big(f(u)\big)v + g\big(f(u)\big)f(v) + f\big(g(u)f(v)\big) \\ &= f^2(u)v + g\big(f(u)\big)f(v) + f\big(g(u)f(v)\big) \qquad (5.69) \end{aligned}$$

Again

$$\begin{aligned} 0 = f^2(uv) = f\big(f(uv)\big) &= f\big(f(u)v + g(u)f(v)\big) \\ &= f(f(u)v) + f\big(g(u)f(v)\big) \text{ since } f \text{ is additive} \\ &= f(u)f(v) + f\big(f(u)\big)g(v) + f\big(g(u)f(v)\big) \\ &= f(u)f(v) + f^2(u)g(v) + f\big(g(u)f(v)\big) \qquad (5.70) \end{aligned}$$

By comparing (5.69) and (5.70), we get

$$f^2(u)v + g\big(f(u)\big)f(v) + f\big(g(u)f(v)\big) = f(u)f(v) + f^2(u)g(v) + f\big(g(u)f(v)\big)$$

$$0 = g\big(f(u)\big)f(v) - f(u)f(v)$$

$$\Big(g\big(f(u)\big) - f(u)\Big)f(v) = 0 \qquad (5.71)$$

By applying Lemma 5.2.7 in (5.71), we get

$$g\big(f(u)\big) - f(u) = 0 \quad \text{ since } f(v) \neq 0$$

$$g\big(f(u)\big) = f(u) \qquad (5.72)$$

Again for $u, v \in U$, we may also write

$$\begin{aligned} 0 = f^2(uv) = f\big(f(uv)\big) &= f\big(f(u)v + g(u)f(v)\big) \\ &= f(f(u)v) + f\big(g(u)f(v)\big) \text{ since } f \text{ is additive} \\ &= f(u)f(v) + f\big(f(u)\big)g(v) + g(u)f\big(f(v)\big) + f\big(g(u)\big)g\big(f(v)\big) \\ &= f(u)f(v) + f^2(u)g(v) + g(u)f^2(v) + f\big(g(u)\big)g\big(f(v)\big) \end{aligned}$$

We have,

$$f(u)f(v) + f\big(g(u)\big)g\big(f(v)\big) = 0 \text{ for all } u, v \in U \tag{5.73}$$

Since $f\big(g(u)\big) = g\big(f(u)\big) = f(u)$ for all $u \in U$

From (5.73) we have,

$$f(u)f(v) + f(u)f(v) = 0$$

$$2f(u)f(v) = 0$$

Since N is 2-torsion free element, we get

$$f(u)f(v) = 0 \tag{5.74}$$

By applying Lemma 5.2.7 in (5.74), we have

$f(u) = 0$ for all $u \in U$ which is a contradiction as $f \neq 0$ on U by Lemma 5.2.6

Thus our assumption $f^2(u) = 0$ is wrong.

Hence, $f^2(u) \neq 0$ on U.

Theorem 5.2.9 (Ali *et al.*, 2016) Let N be a right prime near ring and U be a nonzero semigroup ideal of N. Suppose f is a nonzero semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $f(U) \subseteq Z(N)$, then N is a commutative ring.

Proof. We begin by showing that $(N, +)$ is abelian, which by Lemma 4.1.17 is accomplished by producing

$$z \in Z(N)\backslash\{0\} \text{ such that } z + z \in Z(N)$$

Let a be an element of U such that $f(a) \neq 0$

Then for all $x \in N$, $xa \in U$ and $xa + xa = (x + x)a \in U$, so that

$$f(xa) \in Z(N) \text{ and } f(xa) + f(xa) \in Z(N);$$

hence we need only show that there exists $x \in N$ such that $f(xa) \neq 0$

Suppose this is not the case, so that

$$f\big((xa)a\big) = 0$$

$$f(xa)g(a) + xaf(a) = 0$$

$$xaf(a) = 0 \quad \text{for all } x \in N$$

Since $f(a)$ is not a zero divisor by lemma 4.1.16, we get

$$xa = 0 \quad \text{for all } x \in N$$

$$Na = \{0\}$$

$$a = 0 \quad \text{this is a contradiction.}$$

Therefore, $(N, +)$ is abelian as required.

We are given that $[f(u), x] = 0$ for all $u \in U$ and $x \in N$

Replacing u by uv, we get

$[f(uv), x] = 0$, which yields

$[f(u)v + g(u)f(v), x] = 0$ for all $u, v \in U$ and $x \in N$

Since $(N, +)$ is abelian , this gives

$$\big(f(u)v + g(u)f(v)\big)x = x\big(f(u)v + g(u)f(v)\big)$$

$f(u)vx + g(u)f(v)x = xf(u)v + xg(u)f(v)$ by Lemma 5.2.5

$f(u)vx - xf(u)v = xg(u)f(v) - g(u)f(v)x$

$f(u)vx - f(u)xv = xg(u)f(v) - g(u)xf(v)$ since $f(u)$ and $f(v)$ commutes

$f(u)vx - f(u)xv + g(u)xf(v) - xg(u)f(v) = 0$

$f(u)[v, x] + [g(u), x]f(v) = 0$ for all $u, v \in U$ and $x \in N$ (5.75)

Replacing x by $g(u)$, we obtain

$f(u)[v, g(u)] + [g(u), g(u)]f(v) = 0$

$f(u)[v, g(u)] = 0$ for all $u, v \in U$

By choosing $u \in U$ such that $f(u) \neq 0$ and applying Lemma 4.2.13(ii), we get

$[v, g(u)] = 0$ since $f(u) \neq 0$

Thus, $g(u) \in Z(N)$

It then follows from (5.75) that

$f(u)[v, x] + [g(u), x]f(v) = 0$

$f(u)[v, x] + \big(g(u)x - xg(u)\big)f(v) = 0$

$f(u)[v, x] + (g(u)x - g(u)x)f(v) = 0$ since $g(u) \in Z(N)$

$f(u)[v, x] = 0$ for all $v \in U$ and $x \in N$

$[v, x] = 0$ since $f(u) \neq 0$

Thus, $v \in Z(N)$ for all $v \in U$ implies that $f(v) \in Z(N)$

Hence, $U \subseteq Z(N)$ and N is a commutative ring by Lemma 4.2.15.

Theorem 5.2.10 (Ali *et al*., 2016) Let N be a 2-torsion free zero-symmetric right prime near ring and U be a nonzero semigroup ideal of N. Suppose f is a nonzero semiderivation of N associated with a map g such that $g(U) = U$ and $g(uv) = g(u)g(v)$ for all $u, v \in U$. If $[f(U), f(U)] = \{0\}$, then N is a commutative ring.

Proof. Suppose that $[f(U), f(U)] = \{0\}$, then

$f(u)f(vf(w)) = f(vf(w))f(u)$ for all $u, v, w \in U$ since $f(U) \in Z(N)$

$f(u)\Big(f(v)g(f(w)) + vf\big(f(w)\big)\Big) = \Big(f(v)g(f(w)) + vf\big(f(w)\big)\Big)f(u)$

$f(u)(f(v)g(f(w)) + vf^2(w)) = (f(v)g(f(w)) + vf^2(w))f(u)$ for all $u, v, w \in U$

Then by Lemma 5.2.5, we get

$$f(u)f(v)g(f(w)) + f(u)vf^2(w) = f(v)g(f(w))f(u) + vf^2(w)f(u)$$

Using the fact that $gf = fg$, we get

$$f(u)f(v)f(g(w)) + f(u)vf^2(w) = f(v)f(g(w))f(u) + vf^2(w)f(u) \text{ for all } u, v, w \in U$$

Since $g(U) = U$ and $[f(U), f(U)] = \{0\}$, we obtain

$$f(u)f(v)f(w) + f(u)vf^2(w) = f(v)f(w)f(u) + vf^2(w)f(u)$$

$$f(u)f(v)f(w) + f(u)vf^2(w) = f(v)f(u)\,f(w) + vf^2(w)f(u)$$

$$f(u)f(v)f(w) + f(u)vf^2(w) = f(u)\,f(v)f(w) + vf^2(w)f(u) \text{ for all } u, v, w \in U$$

$$f(u)vf^2(w) = vf^2(w)f(u) \text{ for all } u, v, w \in U \tag{5.76}$$

Replace v by xv, then we get

$$f(u)xvf^2(w) = xvf^2(w)f(u) \text{ for all } u, v, w \in U \text{ and } x \in N$$

$$f(u)xvf^2(w) = xf(u)vf^2(w) \text{ for all } u, v, w \in U \text{ and } x \in N \quad \text{using } (5.76)$$

$$\big(f(u)x - xf(u)\big)vf^2(w) = 0$$

$$[f(u), x]vf^2(w) = 0 \text{ for all } u, v, w \in U \text{ and } x \in N$$

$$[f(u), x]Uf^2(w) = \{0\} \text{ for all } u, v, w \in U \text{ and } x \in N$$

By Lemma 4.2.14, we have either

$$[f(u), x] = 0 \quad \text{or} \quad f^2(w) = 0$$

Since $f^2(U) \neq 0$ by theorem 5.2.8**,** we get

$[f(u), x] = 0$ for all $x \in N$ since $f(u)$ commutes with every elements of N.

Then, $f(u) \in Z(N)$ for all $u \in U$. Thus, $f(U) \subseteq Z(N)$

Hence, N is commutative ring by theorem 5.2.9.

Remark 5.2.11. The primeness condition is necessary in the hypothesis of Theorems 5.2.1 − 5.2.4.

The following example justifies the above remark.

Example 5.2.12 (Boua *et al*., 2014) Let S be a 2-torsion free noncommutative near ring and let us define N and $f, g: N \to N$ by:

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} : x, y, z \in S \right\}$$

$$f\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}$$

Then, N is not prime near ring admitting a nonzero semiderivation f associated with g. Moreover, f satisfies the properties:

$(i)\ f(N) \subseteq Z(N)$

$(ii)\ f(-N) \subseteq Z(N)$

$(iii)\ f([A, B]) = 0$

$(iv)\ f([A, B]) = [A, B]$

$(v)\ f(AoB) = 0$

$(vi)\ f(AoB) = AoB$ for all $A, B \in N$.

However, N is not commutative.

5.3. Semiderivations acting as Homomorphism or as anti- Homomorphism of Near Rings in the Setting of Semigroup Ideals

The main purpose of this subsection establishes similar results in the setting of semigroup ideals of a prime near ring admitting a semiderivation acting as a homomorphism or an anti-homomorphism. We discussed that either semiderivation f is an identity map on semigroup ideal U or a semiderivation f is zero.

The following theorems $(5.3.1 - 5.3.2)$ show that the extensions of derivation on rings with Lie-ideals to semiderivation on near rings in the setting of semigroup ideals which gives the concepts of semiderivation acting as homomorphism or anti-homomorphism on near rings in the setting of semigroup ideals which facilitates either f is an identity or f is zero.

Theorem 5.3.1 (Ali *et al*., 2016) Let N be a zero-symmetric right prime near ring and U be a nonzero semigroup ideal of N. Suppose f is a semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If f acts as a homomorphism on U, then either f is an identity map on U or $f = 0$.

Proof. By the hypothesis

$$f(uv) = f(u)f(v) \text{ for all } u, v \in U$$

$$f(u)g(v) + uf(v) = f(u)f(v) \text{ for all } u, v \in U$$

Replacing u by wu in the above relation, we get

$$f(wu)g(v) + wuf(v) = f(wu)f(v) \text{ for all } u, v, w \in U$$

$$f(w)f(u)g(v) + wuf(v) = f(w)f(u)f(v)$$

$$f(w)f(u)g(v) + wuf(v) = f(w)f(uv) \quad \text{since } f \text{ is homo.}$$

$$f(w)f(u)g(v) + wuf(v) = f(w)(f(u)g(v) + uf(v)) \tag{5.77}$$

Using Lemma 5.2.5, the relation (5.77) yields that

$$f(w)f(u)g(v) + wuf(v) = f(w)f(u)g(v) + f(w)uf(v)$$

$$wuf(v) = f(w)uf(v)$$

$$f(w)uf(v) - wuf(v) = 0$$

$$(f(w) - w)uf(v) = 0 \text{ for all } u, v, w \in U$$

$$(f(w) - w)Uf(v) = \{0\}$$

It follows by Lemma 4.2.14 that either

$$f(U) = \{0\} \text{ or } f(w) = w \text{ for all } w \in U$$

In the first case, $f = 0$ by Lemma 5.2.6

Hence, f is an identity map on U or $f = 0$.

Theorem 5.3.2 (Ali *et al*., 2016) Let N be a zero-symmetric right prime near ring and U be a nonzero semigroup ideal of N. Suppose f is a semiderivation of N associated with a map g such that $g(uv) = g(u)g(v)$ for all $u, v \in U$. If f acts as an anti-homomorphism on U, then either $f = 0$ or N is a commutative ring and f is the identity map on U.

Proof. By the hypothesis

$$f(uv) = f(v)f(u) \text{ for all } u,v \in U$$

$$f(u)g(v) + uf(v) = f(v)f(u) \text{ for all } u,v \in U \tag{5.78}$$

Replacing u by uv in (5.78), we get

$$f(uv)g(v) + uvf(v) = f(v)f(uv) \text{ for all } u,v \in U$$

$$f(v)f(u)g(v) + uvf(v) = f(v)(f(u)g(v) + uf(v))$$

Using Lemma 5.2.5, we get

$$f(v)f(u)g(v) + uvf(v) = f(v)f(u)g(v) + f(v)uf(v)$$

$$uvf(v) = f(v)uf(v) \quad \text{for all } u,v \in U \tag{5.79}$$

Replacing u by ru in (5.79), we get

$$ruvf(v) = f(v)ruf(v) \text{ for all } u,v \in U \text{ and } r \in N \tag{5.80}$$

Using (5.79), the relation (5.80) gives that

$$ruvf(v) = f(v)ruf(v)$$

$$rf(v)uf(v) = f(v)ruf(v)$$

$$f(v)ruf(v) - rf(v)uf(v) = 0$$

$$\big(f(v)r - rf(v)\big)uf(v) = 0$$

$$[f(v),r]uf(v) = 0 \quad \text{for all } u,v \in U \text{ and } r \in N$$

$$[f(v),r]Uf(v) = \{0\}$$

Application of Lemma 4.2.14 yields that either

$$f(v) = 0 \text{ or } [f(v),r] = 0 \text{ and in either case } f(v) \in Z(N)$$

Since from $f(v) \in Z(N)$ for all $v \in U$, we have

$$f(U) \subseteq Z(N)$$

Then N is a commutative ring by theorem 5.2.9.

Now let us comes to the initial step, then

$$f(uv) = f(u)g(v) + uf(v) = f(v)f(u) \text{ for all } u,v \in U$$

Since $f(v) \in Z(N)$, then $f(u)f(v) = f(v)f(u)$ for all $u \in U$

Now from the hypothesis

$$f(uv) = f(u)f(v) \text{ for all } u,v \in U$$

So, f acts as a homomorphism on U.

Hence, f is an identity map on U or $f = 0$ by theorem 5.3.1.

6. SUMMARY AND CONCLUSION

6.1. Summary

This project elaborated the ideas and concepts on semigroup ideals and commutativity of near rings satisfying certain differential identities involving semiderivations. We first discussed the important preliminary concepts, definitions, lemmas and theorems to make the concept clear. Then a typical problem of near ring N with semiderivation have been stated and proved on theorems that deal with sufficient conditions under which N is commutative ring. Some results on commutativity of near rings related to semigroup ideals involving semiderivations were presented. Also, we showed that in some results the primeness, n-torsion free and zero symmetric of near ring N conditions in the setting of semigroup ideals involving semiderivations were sufficient for N to be a commutative ring. Moreover, we discussed some results of semigroup ideals and commutativity of near rings with semiderivations satisfying certain differential identities like:

i. $f(N) \subseteq Z(N)$ for all $x, y \in N$

ii. $f(U) \subseteq Z(N)$

iii. $f(xoy) = [x, y]$ for all $x, y \in N$.

Furthermore, in this project work we discussed semiderivation acting as a homomorphism or anti-homomorphism in the setting of semigroup ideal U of a near ring N to show that either semiderivation f is an identity map on semigroup ideal U or a semiderivation f is zero.

6.2. Conclusion

In this project, we discussed the commutativity of near rings N with semiderivation f was used on N with the existence of an additive map g. It was observed that, a semigroup ideal U of a near rings N determine the commutativity of near rings with semiderivations. Also we conclude that a prime near ring N involving semiderivation satisfying certain differential identities is commutative ring.

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