MODIFIED SIMPLEX METHOD FOR SOLVING LINEAR FRACTIONAL/QUADRATIC BI-LEVEL PROGRAMMING PROBLEM

MSc PROJECT

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APPROVAL SHEET

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I hereby certify that I have read and evaluated this Project entitled **“Modified Simplex Method for Solving Linear Fractional Quadratic Bi-level Programming Problem”** which has been prepared under my guidance by Mohammed Yimer. All feedback given to the student has been incorporated in the project. So, I recommend that it be submitted as fulfilling the project requirements.

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As a member of the Board of Examiners of the MSc Project Open Defense Examination, I certify that I have read and evaluated the Project prepared by Mohammed Yimer and examined the candidate. I recommend that the Project be accepted as fulfilling the Project requirements for the degree of Master of Science in Mathematics (Optimization).

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The author, Mohammed Yimer was born on 14 October, 1995 at Assosa Zone, Benishangul Gumuz Region, Ethiopia, to his father Yimer Legesse and his mother Misaye Geremew. He attended his primary education at Hoha No. 3 primary school. Then after, he joined Assosa secondary school to attend his secondary education. Then he joined Debre Tabor University in 2013 and received Bachelor of Education degree in Mathematics in July 2016. By the year he graduated, he has been employed as an Assistant Mathematics Instructor by Ministry of Education to work at Mekidela Amba University and got direct chance to attend his postgraduate program at Haramaya University, College of Natural and Computational Science, Department of Mathematics and pursued a program of study for MSc degree in Mathematics with Specialization of optimization.

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ABBREVATIONS

BLPP	Bi-level Programming Problem
DM	Decision Maker
KKT	Karush-Kuhn-Tucker
LFBLPP	Linear Fractional Bi-level Programming Problem
LFPP	Linear Fractional Programming Problem
LFPPCC	Linear Fractional Programming Problem with Complementary Constraint
LFQBLPP	Linear Fractional/Quadratic Bi-level Programming Problem
LQBLPP	Linear Quadratic Bi-level Programming Problem
MILFPP	Mixed Integer Linear Fractional Programming Problem
MSM	Modified Simplex Method
NLP	Non-Linear Programming

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ABSTRACT

The main objective of this study was to solve a Linear Fractional/Quadratic Bi-level Programming Problem in which the objective function of the first level (leader) is linear fractional and the lower level (follower) is convex quadratic. The variables associated with both level problems are related by linear constraints. Since the objective of the upper and lower level decision makers are potentially conflicting in nature, a possible transformation of an LFQBLPP into equivalent single level LFPP with complementary constraints are considered by providing KKT necessary and sufficient condition at the lower level. Lagrangian functions have been used to transform the constrained follower problem into unconstrained problem; thereafter modified simplex algorithm was applied to find the solution of an LFPP which satisfies the complementary constraints and hence the optimum solution of an LFPP with complementary constraints determines the optimum solution of the given LFQBLPP. We have stated that for the coefficient of all constraints are negative with the right hand side positive the initial point is infeasible so we can solve that type problem by adding an artificial variable. For LFPP with complementary constraint, optimality conditions were derived based on KKT conditions. The problem was solved analytically and numerical examples were solved to observe the efficiency of the employed method. The proposed method achieves efficient and feasible solution and it is evaluated by comparing with the references. Finally, this study was suggested a further study for searching an algorithm such as genetic algorithm for solving an LFQBLPP and showing the efficiency of the proposed algorithm for solving other kinds of BLPP such as quadratic bi-level programming problem and so on.

Keywords: LFQBLPP, MSM and KKT conditions

1. INTRODUCTION

1.1. Background of the Study

Investigating for and arriving at the best possible decision in any given circumstance is called optimization. The ultimate aim of all such decisions is to maximize the gain (profit) or minimize the cost or loss incurred in a certain process. Thus, optimization can be defined as the maximization or minimization of a function of several variables. This function may be unconstrained or it may be subjected to certain constraints on the variables in the form of equations or inequalities (Nocedal and Wright, 2006).

Optimization is the process of finding the maximum or the minimum value of a function. For the given problem, we formulate a mathematical description called a mathematical problem to represent the situation containing the following components (Boyd and Vandenberghe, 2004):

- ✓ Decision variables: Variables are the quantities we have under our control. We must decide what the best values of the variables are. For this reason, variables are sometimes also called decision variables. The goal of optimization is to find the values of model variables that generate the best value for the objective function, subject to any limiting conditions placed on the variables.
- ✓ Objective function: The objective function is a formula that expresses exactly what we want to optimize. In business-oriented models, this will usually be a profit function we wish to maximize or a cost function we want to minimize.
- ✓ Constraints: Almost without exception, there will be some limit on the values the variables in a model that can assume at least one resource will be limited (e.g., time, raw materials, your department's budget, etc.). These limits are expressed in terms of formulas that are a function of the model's variables. These formulas are referred to as constraints because they constrain the values the variables can take.

A mathematical optimization problem, or just an optimization problem, has the form:

$$\left.\begin{array}{l} max/min \quad f(x) \\ s.t \quad g_i(x) \leq b_i \\ \qquad h_i(x) = 0, \;\; \forall\, i = 1, 2, \ldots, m \end{array}\right\} \tag{1.1}$$

Here the vector $x = (x_1, \dots, x_n)$ is the optimization variable of the problem, the function f: $\mathbb{R}^n \to \mathbb{R}$ is the objective function, the functions $g_i: \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, m,$ and $h_i: \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \dots, m$ are the inequality and equality constraint functions, and the constants $b_1, \dots, b_m$ are the limits, or bounds, for the constraints. For minimization case, x^* is called optimal, or a solution of the problem (1.1), if it has the smallest objective value among all vectors that satisfy the constraints: for any z with $f_1(z) \leq b_1, \dots, f_m(z) \leq b_m$ and $f(z) \geq f(x^*)$. But for maximization case, x^* is called optimal, or a solution of the problem (1.1), if it has the largest objective value among all vectors that satisfy the constraints: for any z with $f_1(z) \geq b_1, \dots, f_m(z) \geq b_m$ and $f(z) \leq f(x^*)$.

A bi-level programing problem (BLPP) is an optimization problem where the feasible set is partly determined through a solution set mapping of a second parametric optimization problem (Dempe, 2002). Bi-level programming problem is a nested optimization problem (*i*.e. the constraint region of the upper-level problem is implicitly determined by optimization problem of the lower level problem) which has two levels in the hierarchy (Sinha *et al.*, 2018). The first level is called the leader and the second level is called the follower which they have their own objective functions and constraints. Bi-level optimization in its original formulations goes back to Stackelberg who introduced a special case of such problems in (1934) when he investigated real market situations. So this particular formulation is a hierarchical game of two players now called Stackelberg game. The formulation of the bi-level optimization problem goes back to Bracken and McGill (1973).

The general BLPP can be formulated as follows:

$$\begin{cases} \underset{x \in X}{max/min}\, F(x, y) \\ \quad where\ y\ solves \\ \underset{y \in Y}{max/min}\, f(x, y) \\ subject\ to\ g(x, y) \leq 0, \end{cases}$$

where $F: \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{m_1},\ f: \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{m_2}$

$$g: \mathbb{R}^{n_1+n_2} \to \mathbb{R}^r \quad with\ n_1, n_2, m_1, m_2, r \in \mathbb{N}$$

For $x \in \mathbb{R}^{n_1}$ fixed, the problem: $\{\underset{y}{max/min}\, f(x, y)\ s.t\ \ g(x, y) \leq 0\}$ is called the lower level or the follower problem parameterized by y, $y \in \mathbb{R}^{n_2}$. F and f are the leader (or higher level) and the follower (or lower level) of the objective functions respectively. The set X and Y place additional constraint on the variables, such as upper and lower bounds or integrality requirements, etc.

If $m_1 = m_2 = 1$, then the functions F and f are scalar-valued; meaning that the higher and lower level decision makers (DM) are optimizing each only one objective. This class of problems is called bi-level single objective optimization problems, or simply bi-level optimization problems. And also if $m_1 > 1$ and/or$m_2 > 1$, the leader and/or follower objective functions are vector valued. We obtain a more general problem called bi-level multi-objective problem. In this case, the upper-level DM and/or the lower level one is optimizing more than one (in general conflicting) objective simultaneously (Dedzo *et al.*, 2012).

The general terminology and basic properties of the BLPP are as follows:

- The relaxed BLPP feasible set (or constraint region):

  $$\Omega = \{(x, y) : g(x, y) \leq 0\}.$$

- For $x \in X$ the inner problem feasible set (follower's solution set, or follower's feasible region):

  $$\Omega(x) = \{y : y \in Y, g(x, y) \leq 0\}.$$

- For$\boldsymbol{x} \in X$, inner rational reaction set (follower's rational reaction set):

  $$R(x) = \{y \in argmin\, f(x, y) : y \in \Omega(x)\}.$$

- The BLPP feasible set (the induced, or inducible region):

  $$IR = \{(x, y) : (x, y) \in \Omega, y \in R(x)\}.$$

For the BLPP to be well-posed, it is generally assumed that Ω is nonempty and compact, and *R(x)*≠ ∅. Using the above terminology, the BLPP can be written as:

$$\begin{aligned} &\min/\max F(x, y) \\ &s.t. (x, y) \in IR. \end{aligned}$$

Definition: A point (x^*, y^*) is said to be optimal to BLPP if

i. $(x^*, y^*) \in IR$ and

ii. $F(x^*, y^*) \leq F(x, y) \quad \forall\, (x, y) \in IR$ for minimization case and

$F(x^*, y^*) \geq F(x, y) \quad \forall\, (x, y) \in IR$ for maximization case.

In this Project, Linear Fractional/Quadratic Bi-level Programming Problem (LFQBLPP) has been considered in which the first level (leader) objective function is linear fractional wherein convexity and differentiability are mandatory as the leader's problem is a general mathematical programming with the point that the common constraint is affine and the second level (follower) problem is convex quadratic. So that the LFQBLPP can be formulated as follows:

$$\begin{cases} \underset{x\in\Omega}{max/min}\, F(x,y) = \frac{a_1 + {a_2}^T x + {a_3}^T y}{b_1 + {b_2}^T x + {b_3}^T y} \\ \qquad\qquad where\ for\ a\ given\ y\ solves \\ \underset{y\in\Omega(x)}{max/min}\, f(x,y) = c^T x + d^T y + (x^T, y^T) Q (x^T, y^T)^T \\ \qquad\qquad subject\ to\ \ g(x,y) = Ax + By \leq r, \end{cases}$$

$where\ a_1, b_1 \in \mathbb{R}, a_2, b_2, c \in \mathbb{R}^{n_1}, a_3, b_3, d \in \mathbb{R}^{n_2}, A \in \mathbb{R}^{m\times n_1}, B \in \mathbb{R}^{m\times n_2}, r \in \mathbb{R}^m, x \in X \subseteq \mathbb{R}^{n_1}, y \in Y \subseteq \mathbb{R}^{n_2}$ and $F(x,y)$,$f(x,y)$ are the objective functions of the leader and the follower, respectively. Also $Q \in R^{n_1+n_2} \times R^{n_1+n_2}$ is symmetric positive semi –definite matrix. The follower problem is convex if $f(x,y)$ and $g(x,y)$ are convex functions in y for any value of x (*i.e.*, if the lower level problem is convex). The advantage of dealing with the convex lower level programming problem is that under an appropriate constraint qualification, the lower level problem can be replaced by its Karush-Kuhn Tucker (KKT) conditions to obtain an equivalent one-level mathematical program. In this Project, in order to solve the LFQBLPP efficiently, the problem has been first changed into an equivalent single level programming problem using KKT conditions, in which the follower problem is replaced by the linear complementarity system. Then, a Modified Simplex Method (MSM) was proposed to solve the equivalent problem.

1.2. Statement of the Problem

The BLPP is an optimization problem in which the constraints are implicitly determined by another optimization problem. In other words, it is a hierarchical optimization problem consisting of two levels. The BLPPs are intrinsically hard because these problems are neither convex nor continuous for the objective function of the upper-level problem is decided by the solution function of the lower-level problem. Bi-level problems are considered very difficult for most existing optimization approaches. Even the simplest case of linear BLPP, where both upper and lower level objectives and all constraints are linear is in general difficult and were shown to be NP-hard (Ben-Ayed and Blair, 1990; Hansen *et al*., 1992). Moreover, even when both levels are convex programming problems, the resultant BLPP can be non-convex (Ben-Ayed, 1993).

Suleiman and Nawkhass (2013) studied on a solving quadratic fractional programming problem by using an MSM. Camp and Scheimberg (2000) have used an MSM for solving bi-level linear programming problems. Hosseini and Kamalabadi (2015) developed two effective approaches based on an MSM and genetic algorithm to solve Linear Quadratic Bi-level Programming Problem (LQBLPP) and Linear Fractional Bi-level Programming Problem (LFBLPP). Calvete and Galé (2004) considered LFQBLPP, in which the first level objective function is linear fractional, the second level objective function is quadratic and the common constraint is linear. Its optimality conditions are derived based on KKT conditions and duality theory.

Arora and Arora (2011) proposed to solve an LFQBLPP in which the upper level objective function is linear fractional and the lower level is quadratic. On applying the KKT conditions to the lower level problem and by making use of the complementarity condition and the upper level objective function a fractional programming problem is formed. The optimum solution of this problem determines the optimum solution of the given BLPP by using modified simplex algorithm. They have been considered only the maximization case. In order to extend this work by taking an LFQBLPP in which the objective function of the upper level is linear-fractional and the objective function of the lower level is convex quadratic and all the constraint are linear we considered both maximization and minimization case. The constrained second level programming problem has also been transformed into the unconstrained problem by using the Lagrangian function method.

This Project has tried to answer the following questions:

1. How to describe the LFQBLP problem?
2. How the LFQBLPP could be converted into single level programming problem?
3. To what extent an MSM was efficient for solving LFQBLPP and how this efficiency can be figured out?

1.3. Objectives

The main objective of this Project was to solve LFQBLPP using an MSM. The study was further intended to explore the following specific objectives:

- ✓ To describe LFQBLP problems.
- ✓ Convert the LFQBLPP into single level programming problem using KKT condition by which Lagrangian function is used to transform a constrained second level programming problem into the unconstrained problem.
- ✓ Showing the efficiency of an MSM for solving LFQBLPP.

2. REVIEW OF RELATED LITERATURE

2.1. Bi-level programming problem

Bi-level programming dates back (Stackelberg, 1934) in the monograph and he formulated a hierarchical game of two players now called Stackelberg game. The formulation of the BLPP goes back to (Bracken and McGill, 1973). One possibility to investigate bi-level programs is to transform them into single-level (or ordinary) optimization problems. In the first years, linear BLPP, where all the involved functions are affine (linear) were usually transformed making use of linear programming duality or, equivalently, the KKT conditions for linear programming.

Alemayehu and Arora (2001) developed an algorithm to find the optimal integer solution for bi-level programming problem in which both the objective functions were linear fractional and the variables take non-negative integral values. They also considered a Mixed (0, 1) Linear Fractional Programming problem. A new approach to obtain the solution of Mixed (0, 1) Linear Fractional Programming problem was developed by reformulating Mixed (0, 1) Linear Fractional Programming problem into a bi-level Fractional Programming Problem. Arora (2001) presented an algorithm to solve a BLPP, in which the leader's and the follower's objective functions were both quadratic. They developed a quadratic programming problem equivalent to the BLPP. The leader's and follower's objective functions were both concave. The follower's problem was replaced by the corresponding KKT necessary and sufficient optimality conditions to formulate a quadratic programming problem.

Hejazi *et al.* (2002) present a method based on a genetic algorithm approach. To this end, KKT conditions for the second level problem are derived and then the BLPP is transferred into a single level problem with complementary constraints. Then, the genetic algorithm is employed to solve the transferred problem. Guang-Min *et al*. (2002) used a genetic algorithm method for solving a convex quadratic bi-level programming problem. Thus, a convex quadratic bi-level problem is transformed into a single level problem by applying KKT conditions, and then an efficient method based on genetic algorithm has been proposed for solving the transformed problem. They showed the algorithm is effective in practice by taking numerical experiments on several literature problems.

Mishra and Ghosh (2006) proposed an interactive fuzzy programming method for obtaining a satisfactory solution to a Bi-Level Quadratic Fractional Programming problem with two DMs interacting with their optimal solutions. They derived a satisfactory solution to update the satisfactory level of the DM at the upper level with consideration of the overall satisfactory balance between both levels. Calvete *et al.* (2008) developed a genetic algorithm for the linear bi-level problem in which both objective functions are linear and the common constraint region is a polyhedron. This genetic algorithm can also be used for solving quasi-concave BLPP provided that the second level objective function is linear.

Harbavi (2008) proposed a global optimization approach for the solution of Quadratic Bilevel programming problem using of Taylor Series. First, he transforms both objectives functions by using 1st order Taylor Polynomial Series. Here, the Taylor Series obtains polynomial objective functions which are equivalent to quadratic objective functions. Thus, Bilevel Quadratic problem can be reduced into a single function level are classed as upper-level and lower-level, and they are weighted with respect to their classes before Taylor series approach united levels by using their weights.

Yue *et al.* (2013) developed an algorithm to solve a bi-level programming problem in which the leader's and the follower's objective functions were both quadratic and convex. They replaced the follower's problem by the corresponding KKT necessary and sufficient optimality conditions to formulate a quadratic programming problem which was equivalent to the given problem. More recently, Dempe *et al.* (2013) presented for the optimistic formulation of a bi-level optimization problem with multi-objective lower level problem, necessary optimality conditions by considering the scalarization approach for the lower level multi-objective program and transforming the problem in to a scalar objective optimization problem with inequality constraints by means of the optimal value reformulation.

Rakshit and Kumar (2016) proposed a new model of quadratic fractional programming problem where our purpose is to study the quadratic fractional programming problem through fuzzy goal programming procedure by utilizing the bi-level linear programming. They construct two bi-level quadratic programming problems from one bi-level quadratic fractional programming problem by separating the numerator and denominator in the fractional objective function of each decision maker.

2.2. Linear Fractional/Quadratic Bi-level Programming Problem

Malhotra and Arora (2001) proposed linear fractional-quadratic bi-level programming problem in which the leader objective function is linear fractional and the follower objective function is quadratic, the leader's and follower's variables are related by linear constraints. The derivations of optimality conditions are based on KKT conditions and the duality theory. So that instead of solving the original LFQBLPP they solve the Linear Fractional Programming Problem (LFPP) and the optimal solution of the original problem can be achieved at one of the extreme points of a convex polyhedral formed by the new feasible region.

Calvete and Galé (2004) proposed the bi-level linear fractional/linear programming problem, in which the objective function of the first-level is linear fractional, the objective function of the second level is linear, and the common constraint region is a polyhedron. By taking into account the relationship between the optimization problem of the second level and it's dual, a global optimization approach is proposed that uses an exact penalty function based on the duality gap of the second-level problem. They have considered the LFQBLPP, in which the first level objective function is linear fractional, the second level objective function is quadratic and the common constraint region is a polyhedron. For this problem, optimality conditions are derived based on KKT conditions and duality theory.

Wang *et al*. (2008) have also used KKT conditions for the lower level programming problem to transform BLPP into a single level programming which can be also simplified to linear programming by the chromosome according to the rule. Thus, in their proposed genetic algorithm, only the linear programming is solved by the simplex method to obtain the feasibility and fitness value of the chromosome. Etoa (2011) used the KKT optimality conditions of the lower level problem to obtain a non-smooth optimization problem known to be a mathematical program with equilibrium constraints; the complementary conditions of the lower level problem are then appended to the upper-level objective function with a classical penalty. The complementary conditions are relaxed by mean of semi-smooth equations using Fisher–Burmeister functional. Then, using a quadratic sequential programming method, we solve a series of smooth, regular problems that progressively approximate the non-smooth problem.

Maachou and Moulaï (2015) have found the optimality conditions and a solution procedure to solve a bi-level quadratic fractional-quadratic programming problem in which the leader's objective is quadratic fractional and the follower's objective is quadratic. The variables associated with both the level problems are related by linear constraints. The proposed method is based on KKT conditions and a related bi-level linear fractional-quadratic problem is constructed in which the leader's objective is linear fractional and the follower's objective is quadratic in order to obtain an optimal solution of a bi-level quadratic fractional-quadratic programming problem.

2.3. Modified simplex method

Onal (1993) proposed a Modified Simplex Approach to solve a bi-level linear programming problem. The inner problem was represented by its KKT conditions and the complementary slackness equations were moved to the outer objective as a penalty function. The global optimum of the resulting quadratic program was reached by the iterative application of a modified simplex algorithm.

Campêlo and Scheimberg (2000) studied an MSM for solving bi-level linear programming problems by point out some problems in its theoretical analysis. Moreover, the algorithm proposed may not find a global solution as it is claimed. They give some examples in order to illustrate these remarks. Xu *et al.* (2000) used an MSM for the optimization of a new flow injection-kinetic system for the spectrophotometric determination of intravenous with m-acetylchlorophosp- honazo, which has for the first time been used as a chromogenic reagent in the quantitative analysis of this element. An orthogonal array design is utilized to design the experimental protocol, in which six variables are varied simultaneously, and obtain the initial simplex using 25 experiments. A modified simplex method is applied to continuously optimize the data of the orthogonal array design; the search for optimum conditions of 6 variables using the MSM required only 25 experiments.

A modified simplex method has been applied, for the first time, to determine compounds by a luminescence technique. The method was based on the optimization of chemical and instrumental variables affecting phosphorescence using a geometric simplex in two and three dimensions of space, respectively (Pulgarin *et al*, 2002). Zhao *et al*. (2009) used an MSM by modifying the well-known Nelder-Mead simplex search method for the unconstrained optimization problem. Instead of moving all *n* simplex vertices at once in the direction of the best vertex, our "shrink" step moves them in the same direction but one by one until improvement is obtained. In addition, for solving non-convex problems, they simply restart the so modified Nelder-Mead method by constructing an initial simplex around the solution obtained in the previous phase. They repeat restarts until there is no improvement in the objective function value.

The modified convex simplex method is used to solve the infinite LFPP to obtain the global convergent solution of the original bi-level linear fractional-linear programming by replacing the lower level problem by its dual gap equaling to zero (Wang *et al*, 2010). Tao (2012) discussed the sensitivity analysis of linear BLPP based on an MSM while the value coefficient of the lower level decision contains parameter and studies the conditions of the optimal solution while parameter continues (or perturbation) in a definite direction. The result provides the decision basis for the decision maker.

Suleiman and Nawkhass (2013) define a new MSM to solve quadratic fractional programming problem and suggested an algorithm for it. The algorithm of the usual simplex method is also reported. The special case for this problem was solved by Converting objective function to the pseudo affinity of quadratic fractional functions to a linear programming problem to be solved by the simplex method. Then the result is compared with a result, which obtained by the new MSM. This method was demonstrated by numerical example to confirm the technique is valid or not, consequently reliable results have been found. Sulaiman *et al*. (2014) used an MSM for solving the transformed multi-objective LFPP to single-Objective LFPP, through a new method using arithmetic average and new arithmetic average technique.

Ghadle and Pawar (2015) introduced Wolfe’s MSM that is easy to solve quadratic programming problem concern with a non-linear programming problem. In linear programming models, the characteristic assumption is the linearity of the objective function and constraints. Although this assumption holds in numerous practical situations, yet we come across many situations where the objective function and some or all of the constraints are non-linear functions. The non-linearity of the functions makes the solution of the problem much more involved as compared to Linear Programming Problems and there is no single algorithm like the simplex method, which can be employed to solve efficiently all nonlinear programming problems.

Hosseini and Kamalabadi (2015) are also attempted to develop two effective approaches, one based on MSM and the other based on the genetic algorithm for solving the LQBLPP and LFBLPP. They use to obtain efficient upper bound and lower bound we employ the KKT conditions for transforming the LQBLPP into a single level problem and also penalty function method is used to transform the single problem into unconstraint problem

Santos *et al*. (2016) used a modified simplex algorithm to optimize a system of preconcentration for cadmium determination in samples of sediments, cigarettes, and food using flame atomic absorption spectrometry. The preconcentration system is based on the sorption of cadmium in a minicolumn packed with Amberlite XAD-2 resin functionalized with 3, 4-dihydroxybenzoic acid (DHB).

3. MATERIALS AND METHODS

This chapter outlined the methods that have been used to succeed the main and specific objectives of the study, and materials used in the study. Source in the web and libraries were used to collect all the pieces of information about solving LFQBLPP using an MSM and recorded subsequently. Specifically,

- ✓ Relevant journals and books are addressed to gather information about Solving LFQBLPP using an MSM.
- ✓ The collected informations were analyzed and arranged to keep coherence.
- ✓ Important concepts, definitions, examples and theorems have been discussed in order to clear the idea.
- ✓ KKT condition was used to transform the LFQBLPP into LFPP with complementary constraint.
- ✓ Lagrangian function has been employed to transform a constrained second level programming problem into the unconstrained problem.
- ✓ Using modified simplex algorithm the transformed LFPP with complementary constraint was solved.

4. PRELIMINARIES

In this chapter, definitions, theorems and concepts which are related to the study of solving LFQBLPP using an MSM are dealt, where the upper-level programming problem is linear fractional and the lower level programming problem is convex quadratic and also all the constraint is linear.

4.1. Basic definitions and concepts

Definition 4.1 (Singh, 2012): A set K $\subset \mathbb{R}^n$ is said to be convex if either K=$\emptyset$ or, whenever we take two points in K, the segment that connects them is entirely connected in K. mathematically, K is a convex set if for any two points x_1, x_2 in K, then $\lambda x_1 + (1-\lambda)x_2$ is also in K for any $\lambda \in [0,1]$.

Definition 4.2 (Singh, 2012): Let $K \subseteq \mathbb{R}^n$be a convex set, and let $f: K \to \mathbb{R}$. Then $f(x)$ is convex on K if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2 \in K \, and \, \lambda \in [0,1]$. If

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2).$$

For all $x_1, x_2 \in K, x_1 \neq x_2, and\ \lambda \in (0,1),$ then we say that $f(x)$ is strictly convex. And f is concave if $-f$ is convex that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2).$$

For an affine function, we always have equality in the above definitions, so all affine (and therefore also linear) functions are both convex and concave. Conversely, any function that is convex and concave is affine.

4.2. Linear Fractional Programming Problem

In mathematical optimization, the LFPP is a generalization of linear programming. Whereas the objective functions in linear programming problems are linear functions, the objective function in an LFPP is a ratio of two linear functions. A linear program can be regarded as a special case of an LFPP in which the denominator is the constant function one. Formally, an LFPP is defined as the problem of maximizing (or minimizing) a ratio of affine functions over a polyhedron,

$$\begin{cases} max/min \ \frac{c^T x + \alpha}{d^T x + \beta} \\ subject\ to\ Ax \leq b, \end{cases}$$

where $x \in \mathbb{R}^n$ represents the vector of variables to be determined, $c, d \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are vectors of (known) coefficients, $A \in \mathbb{R}^{m\times n}$ is a (known) matrix of coefficients and $\alpha, \beta \in \mathbb{R}$ are constants. The constraints have to restrict the feasible region to $\{x|d^T x + \beta > 0\}$, *i.e.* the region on which the denominator is positive. Alternatively, the denominator of the objective function has to be strictly negative (Biswas *et al.*, 2017).

4.3. Quadratic programming problem

Quadratic programming is the simplest constrained nonlinear optimization problem. A special case of the Non-linear Programming (NLP) arises when the objective function f is quadratic and the constraints are linear in $x \in \mathbb{R}^n$. Such an NLP is called a Quadratic Programming problem. Its general form is:

$$\begin{cases} max/min\ \ c^T x + \frac{1}{2} x^T Q x \\ \quad subject\ to\ Ax \le b \\ \qquad\quad x \ge 0, \end{cases}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $Q \in \mathbb{R}^{n\times n}$ is a positive definite or positive semi-definite symmetric square matrix (Meyer, 2013).

Example (Hillier and Lieberman, 2001): The following problem is a quadratic programming problem.

$$\begin{cases} max/min\, f(x_1, x_2) = 15x_1 + 30x_2 + 4x_1x_2 - 2{x_1}^2 - 4{x_2}^2 \\ \qquad subject\ to\ x_1 + 2x_2 \le 30 \\ \qquad\qquad x_1 \ge 0, x_2 \ge 0. \end{cases}$$

In this case, $c = \begin{bmatrix} 15 \\ 30 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Q = \begin{bmatrix} -4 & 4 \\ 4 & -8 \end{bmatrix}$, $A = [1, 2]$, $b = [30]$.

Note that $x^T Q x = [x_1\ x_2] \begin{bmatrix} -4 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2] \begin{bmatrix} -4x_1 + 4x_2 \\ 4x_1 - 8x_2 \end{bmatrix}$

$= -4{x_1}^2 + 4x_2x_1 + 4x_1x_2 - 8{x_2}^2$, multiplying by $\frac{1}{2}$ gives

$\frac{1}{2} x^T Q x = -2{x_1}^2 + 4x_1x_2 - 4{x_2}^2.$

Theorem (Freund, 2004): The function $f(x) = \frac{1}{2} x^T Q x + c^T x$ is a convex function if and only if Q is symmetric and positive semi-definite.

Proof: ($\Rightarrow$)Suppose that Q is not symmetric and positive semi-definite.

Then there exists a non-zero column r such that $r^TQr < 0$. Let $x = \theta r$. Then $f(x) = f(\theta r) = \frac{1}{2}\theta^2 r^TQr + \theta c^Tr$ is strictly concave function on the subset $\{x|x = \theta r\}$, since $r^TQr < 0$.

Thus $f(x)$ is not a convex function. Hence, by contrapositive we conclude that if $f(x)$ is convex function then Q is symmetric and positive semi-definite.

(⇐) Let Q is symmetric positive definite. For all $\lambda \in [0,1]$, and for all x, y

$$
\begin{aligned}
f(\lambda x + (1-\lambda)y) &= f(y + \lambda(x-y)) \\
&= \frac{1}{2}\big(y + \lambda(x-y)\big)^T Q\big(y + \lambda(x-y)\big) + b^T(y + \lambda(x-y)) \\
&= \frac{1}{2}y^TQy + \lambda(x-y)^TQy + \frac{1}{2}\lambda^2(x-y)^TQ(x-y) + \lambda c^Tx + (1-\lambda)c^Ty \\
&\leq \frac{1}{2}y^TQy + \lambda(x-y)^TQy + \frac{1}{2}\lambda(x-y)^TQ(x-y) + \lambda c^Tx + (1-\lambda)c^Ty \\
&= \frac{1}{2}\lambda x^TQx + \frac{1}{2}(1-\lambda)y^TQy + \lambda c^Tx + (1-\lambda)c^Ty \\
&= \lambda f(x) + (1-\lambda)f(y).
\end{aligned}
$$

Hence $f(x)$ is a convex function (by the definition of convexity of a function).

Corollary (Freund, 2004): Let the function $f(x) = \frac{1}{2}x^TQx + c^Tx$.

i. $f(x)$ is strictly convex if and only if Q is positive definite.

ii. $f(x)$ is strictly concave if and only if Q is negative definite.

iii. $f(x)$ is neither convex nor concave if and only if Q is indefinite.

Example (Hillier and Lieberman, 2001): Show that the following quadratic function is convex.

$$f(x_1, x_2) = -15x_1 - 30x_2 - 4x_1x_2 + 2{x_1}^2 + 4{x_2}^2$$

Solution: using our standard form $\frac{1}{2}x^TQx + c^Tx$, we proceed as follows:

$c = [15, 30]$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Q = \begin{bmatrix} 4 & -4 \\ -4 & 8 \end{bmatrix}$ is constructed as follows:

- ✓ For squared terms like $c{x_i}^2$, write $2c$ in the ii position of Q. So that $2{x_1}^2$ term in our example appears as 4 in the $1-1$ position of Q.
- ✓ For quadratic term involving a pair of variables like cx_ix_j, write c in both the ij position and ji position of Q. So the $-4x_1x_2$ term in our example objective function appears as two terms; -4 in the $1-2$ position and also -4 in 2-1 position of Q.

- ✓ The linear term in our example like $-15x_1 - 30x_2$ do not appear in Q at all, they appear in the standard linear part of the objective function like $c^T x$.

So to show convexity, it is enough to show that $x^T Q x \geq 0$, therefore

$$x^T Q x = [x_1, x_2]\begin{bmatrix} 4 & -4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [(4x_1 - 4x_2), (-4x_1 + 8x_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$= 4{x_1}^2 - 8x_1x_2 + 8{x_2}^2$ (Dividing both sides by 4) we get:

$= {x_1}^2 - 2x_1x_2 + 2{x_2}^2 = (x_1 - x_2)^2 + {x_2}^2 \geq 0, \forall\, x_1, x_2$ (Since the sum of two nonnegative terms are always non-negative). Hence a matrix Q is symmetric and positive semi-definite. So a quadratic function $f(x_1, x_2)$ is convex function.

Definition 4.4. (Nonlinear programming) (Bertsekas, 1997): A problem of equalities and inequalities collectively termed constraints over a set of unknown real variables along with an objective function to minimize or maximize, where some of the constraints and/or the objective functions are nonlinear is known as NLP.

4.4. Karush-Kuhn-Tucker Condition

The KKT optimality conditions are fundamental for many algorithms for constrained optimization problems. They can also be called first-order conditions because in their derivation properties of the gradients are used. The Lagrange multipliers can be used to develop optimality criteria for the equality constrained optimization problems. Kuhn and Tucker extend this theory to include equality and inequality constrained problem (Josheski and Gelova, 2013).

Now consider the general NLP:

$$\textbf{P:} \begin{cases} max/min\ \ f(x) \\ subject\ to\ g_i(x) \leq 0, \ \ i = 1,2, \dots, m. \\ \qquad\qquad\quad h_j(x) = 0, \ \ j = 1,2, \dots, l. \\ \qquad\qquad\quad x \geq 0, \end{cases}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Theorem (Kuhn and Tucker, 2014): Let $f: \mathbb{R}^n \to \mathbb{R}$ be a differentiable objective function, $g_i: \mathbb{R}^n \to \mathbb{R}$ be differentiable constraint functions for $i = 1, 2, 3, \dots, m$ and $h_j: \mathbb{R}^n \to \mathbb{R}$ be differentiable constraint functions for$j = 1, 2, 3, \dots, l$. If $x^* \in \mathbb{R}^n$ is an optimal point satisfying an appropriate regularity condition for the above optimization problem **P**, then there exists $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$ and $(\mu_1, \mu_2, \dots, \mu_l) \in \mathbb{R}^l$ so that:

Primal feasibility: $\begin{cases} g_i(x^*) \leq 0, \ \ for\ i = 1,2, \dots, m \\ h_j(x^*) = 0, \ \ for\ j = 1, 2, \dots, l \end{cases}$

Dual feasibility: $\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{l} \mu_j \nabla h_j(x^*) = 0 \\ \lambda_i \geq 0, \ \ for\ i = 1, 2, \dots, m \\ \mu_j \in \mathbb{R}, \ \ for\ j = 1, 2, \dots, l \end{cases}$

complementary constraint: $\{\lambda_i g_i(X^*) = 0 \ \ for\ i = 1, 2, \dots, m,$ then x^* is a global maximizer for **P**. where, $(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $(\mu_1, \mu_2, \dots, \mu_l)$ are Lagrangian multipliers.

Jensen and Bard (2003) have studied that excluding the non-negativity condition, the Lagrangian function for the problem P is

$$L(x, \lambda, \mu) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x),$$

where λ and μ are m −dimentional and l −dimentional row vector respectively. Thus, the Kuhn-Tucker conditions for a local maximum are as follows:

$$\frac{\partial L}{\partial x_k} \leq 0, k = 1, 2, \dots, n \qquad \nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{j=1}^{l} \mu_j \nabla h_j(x) \leq 0 \tag{4.1}$$

$$\frac{\partial L}{\partial \lambda_i} \geq 0, \ i = 1, 2, \dots, m \qquad g_i(x) \geq 0 \tag{4.2}$$

$$\frac{\partial L}{\partial \mu_j} = 0, \ j = 1, 2, \dots, l \qquad h_j(x) = 0 \tag{4.3}$$

$$x_k \left(\frac{\partial L}{\partial x_k}\right) = 0, k = 1, 2, \dots, n \qquad x^T\left(\nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{j=1}^{l} \mu_j \nabla h_j(x)\right) = 0 \tag{4.4}$$

$$\lambda_i \left(\frac{\partial L}{\partial \lambda_i}\right) = 0, \ i = 1, 2, \dots, m \qquad \lambda_i\big(g_i(x)\big) = 0 \tag{4.5}$$

$$x_k \geq 0, k = 1, 2, \dots, n \qquad x_k \geq 0 \tag{4.6}$$

$$\lambda_i \geq 0, \ i = 1, 2, \dots, m \qquad \lambda_i \geq 0 \tag{4.7}$$

$$\mu_j \geq 0, \ j = 1, 2, \dots, l \qquad \mu_j \geq 0 \tag{4.8}$$

To put all the above conditions in more manageable forms we introduce the non-negative slack variables $u_k \in \mathbb{R}^n$ into (4.1) and the non-negative surplus variables $v_i \in \mathbb{R}^m$ into (4.2) to obtain the equation $\nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{j=1}^{l} \mu_j \nabla h_j(x) + u_k = 0$, and $g_i(x) - v_i = 0$. thus, the KKT conditions can now be written as follows:

$$\begin{cases}\nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{j=1}^{l} \mu_j \nabla h_j(x) + u_k = 0 \\ \quad g_i(x) - v_i = 0 \\ \quad h_j(x) = 0 \\ x^T\left(\nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{j=1}^{l} \mu_j \nabla h_j(x)\right) = 0 \\ \quad \lambda_i\left(g_i(x)\right) = 0 \\ \qquad x_k \geq 0, \lambda_i \geq 0, \mu_j \geq 0, u_k \geq 0, v_i \geq 0 \end{cases}$$

The first three expressions are linear equalities, the fourth and fifth restricts complementary slackness and the sixth one prescribes all the variables to be nonnegative. The KKT conditions for the local minimum are the same as above except the inequality sign that means for minimization case the inequality sign in equation (4.1) is (≥) and in equation (4.2) is (≤) therefore we add the slack variable in equation (4.2) and the surplus one in (4.1).

Example (Singh, 2012): Find the KKT conditions for the following programming problem.

$$\begin{cases} max/min - {x_1}^2 - {x_2}^2 - {x_3}^2 + 4x_1 + 6x_2 \\ \quad subject\ to\ x_1 + x_2 \leq 2 \\ \qquad\qquad 2x_1 + 3x_2 \leq 12 \\ \qquad\qquad x_1, x_2, x_3 \geq 0. \end{cases}$$

Solution: It is enough to show the minimum of the problem only.

Let $f(x) = -{x_1}^2 - {x_2}^2 - {x_3}^2 + 4x_1 + 6x_2$, $g_1(x) = x_1 + x_2 - 2$, $g_2(x) = 2x_1 + 3x_2 - 12$, $x_1, x_2, x_3 \geq 0$, thus the condition is obtained by the partial derivative of the Lagrangian function where, the Lagrangian function $L(x, \lambda, s) = f(x) + \lambda_1(g_1(x) + s_1) + \lambda_2(g_2(x) + s_2)$ where, $s = (s_1, s_2), \lambda = (\lambda_1, \lambda_2)$ and s_1, s_2 being slack variables and λ_1, λ_2 are Lagrangian multipliers. The KKT conditions are given by:

$$\begin{cases} -2x_1 + 4 + \lambda_1 + 2\lambda_2 - u_1 = 0 \\ -2x_2 + 6 + \lambda_1 + 3\lambda_2 - u_2 = 0 \\ -2x_3 - u_3 = 0 \\ x_1 + x_2 + v_1 = 2 \\ 2x_1 + 3x_2 + v_2 = 12 \\ \lambda_1 v_1 = 0, \lambda_2 v_2 = 0 \\ u_1 x_1 = 0, u_2 x_2 = 0, u_3 x_3 = 0 \\ x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{cases}$$

Feasible solution set: A set of values of the decision variables $x_1, x_2, x_3, \dots, x_n$ which satisfy all the constraints and also the non-negativity condition is called the feasible solution set of the linear programming problem.

Feasible solution: Each element of the feasible solution set is called a feasible solution. A feasible solution is a solution which satisfies all the constraints and also the non-negativity conditions of the linear programming problem.

4.5. Simplex method

Many different methods have been proposed to solve linear programming problems, but simplex method has proved to be the most effective. This Method is applicable to any problem that can be formulated in terms of linear objective function, subject to a set of linear constraints. Often, this method is termed Dantzig's simplex method, in honour of the mathematician who devised the approach (Nash, 2000).

The general steps of the simplex method are as follows:

(1) Start with an initial extreme point of the feasible solution set.

(2) Improve the initial solution if possible by finding another extreme point of a feasible solution set with a better objective function value. At this step, the simplex method implicitly eliminates from consideration all those extreme points of the feasible solution set whose objective function values are worse than the present one. This makes the procedure more efficient than the enumeration method.

(3) Continue to find the better extreme point of the feasible solution set, improving the objective function value at every step.

(4) When a particular extreme point of feasible solution set cannot be improved further, it becomes an optimal solution and the simplex method terminates.

There are a number of computational problems that may arise during the actual application of the simplex method for solving a linear programming problem. In this section, we discuss some complication that may occur (Dantzig and Thapa, 2006).

1. Ties in the selection of the entering variable:

In a maximization problem, the non-basic variable with the most negative coefficient in the objective function is chosen to enter into the basis. In case there exists more than one variable with the same most negative value in the objective function then, we have a tie for selecting the entering variable.

2. Ties in the minimum ratio rule and degeneracy:

While applying the minimum ratio rule it is possible for two or more constraints to give the same least ratio value. This result in a tie for selecting which basic variable should leave the basis. This complication causes degeneracy in a basic feasible solution.

Degenerate basic feasible solution: A basic feasible solution is said to be a degenerate basic feasible solution if at least one basic variable in the solution equals zero. When there exist degeneracy in a feasible solution we can't say that simplex method will be terminated in a finite number of iterations, because this phenomenon causes that we obtain a new basic feasible solution in the next iteration which has no effect on the objective function value.

5. RESULT AND DISCUSSION

5.1. Linear Fractional/Quadratic Bi-Level Problem

Linear fractional/quadratic bi-level programming problem is a particular type of NLP in which the leader objective function is the ratio of two linear programming problem and the follower objective function is quadratic and the common constraint is linear. It is shown that the given LFQBLPP can be solved by an LFPP. The technique used is to replace the follower's problem by the corresponding KKT necessary and sufficient optimality conditions. An alternate representation of the original problem LFQBLPP is made by appending these conditions to the leader's constraint set. The resulting problem becomes linear fractional programming problem with complementary constraint (LFPPCC). It is shown that the optimality condition of the LFPP is an extreme point satisfying the complementary condition. Without loss of generality, we considered the minimization and maximization problems in our discussion. We provide two equivalent formulations for a general LFQBLPP in equations 5.1 and 5.2 below:

5.1.1. Mathematical Formulation of Linear Fractional/Quadratic Bi-Level Problem

Using the common notation in BLPP, the LFQBLP problem can be written as follows:

$$\begin{cases} \max\limits_{x \in \Omega} F(x,y) = \frac{a_1 + {a_2}^T x + {a_3}^T y}{b_1 + {b_2}^T x + {b_3}^T y} \\ \qquad\qquad \textit{where for a given } y \textit{ solves} \\ \max\limits_{y \in \Omega(x)} f(x,y) = c^T x + d^T y + (x^T, y^T) Q (x^T, y^T)^T \\ \qquad \textit{subject to} \quad (x,y) \in \Omega, \end{cases} \tag{5.1}$$

where $\Omega = \{(x,y) \in \mathbb{R}^{n_1+n_2} \colon Ax + By \le r \,; x, y \ge 0\}$; $a_1, b_1 \in \mathbb{R}$; $a_2, b_2, c \in \mathbb{R}^{n_1}$; $a_3, b_3, d \in \mathbb{R}^{n_2}$; $A \in \mathbb{R}^{m \times n_1}$; $B \in \mathbb{R}^{m \times n_2}$; $r \in \mathbb{R}^m$; $x \in X \subseteq \mathbb{R}^{n_1}$; $y \in Y \subseteq \mathbb{R}^{n_2}$ and Q is an $((n_1 + n_2) \times (n_1 + n_2))$ dimensional symmetric negative semi-definite matrix (*i.e.*, $Q \in R^{n_1+n_2} \times R^{n_1+n_2}$). With

$$Q = \begin{bmatrix} Q_2 & {Q_1}^T \\ Q_1 & Q_0 \end{bmatrix},$$

where $Q_0 \in \mathbb{R}^{n_2 \times n_2}, Q_1 \in \mathbb{R}^{n_2 \times n_1}$ *and* $Q_2 \in \mathbb{R}^{n_1 \times n_1}$ are matrices of conformal dimensions. In order to ensure that the problem (5.1) is well-posed, it is assumed that the common constraint region $\Omega = \{(x,y) \colon Ax + By \le r, x, y \ge 0\}$ is non-empty and bounded. In addition, it is also assumed that $b_1 + {b_2}^T x + {b_3}^T y > 0, \forall (x,y) \in \Omega$. If this is not so, it suffices to consider the linear fractional objective function as $\frac{-(a_1+a_2{}^Tx+a_3{}^Ty)}{-(b_1+b_2{}^Tx+b_3{}^Ty)}$. Define, $\Omega(x) = \{y \in \mathbb{R}^{n_2}\ ; (x,y) \in \Omega\}$ gives the feasible region of the lower level problem, for any arbitrary value of x. And also since $Q = \begin{bmatrix} Q_2 & Q_1{}^T \\ Q_1 & Q_0 \end{bmatrix}$, the second level (follower) objective function in (5.1), for a given x is formulated as:

$$\begin{aligned}
\min_{y\in\Omega(x)} f(x,y) &= c^Tx + d^Ty + (x^T, y^T)Q(x^T, y^T)^T \\
&= c^Tx + d^Ty + (x^T, y^T)\begin{pmatrix} Q_2 & Q_1{}^T \\ Q_1 & Q_0 \end{pmatrix}(x^T, y^T)^T \\
&= c^Tx + d^Ty + (x^TQ_2 + y^TQ_1, x^T\ Q_1{}^T +\ y^TQ_0)\begin{pmatrix} x \\ y \end{pmatrix} \\
&= c^Tx + x^TQ_2x + (d + 2Q_1x)^Ty + y^TQ_0y.
\end{aligned}$$

Therefore problem (5.1) is transformed into the following equivalent formulation:

$$\begin{cases}
\max\limits_{x\in\Omega} F(x,y) = \frac{a_1+a_2{}^Tx+a_3{}^Ty}{b_1+b_2{}^Tx+b_3{}^Ty} \\
\qquad\qquad where\ y\ solves \\
\max\limits_{y\in\Omega(x)} f(x,y) = c^Tx + x^TQ_2x + (d + 2Q_1x)^Ty + y^TQ_0y \\
\qquad\quad subject\ to\ Ax + By \le r \\
\qquad\qquad\quad x, y \ge 0.
\end{cases}$$

For each value of $x \in \Omega$, the second level decision maker solves the following quadratic programming problem:

$$\begin{cases}
\max\limits_{y\in\Omega(x)} f(x,y) = c^Tx + d^Ty + (x^T, y^T)Q(x^T, y^T)^T \\
\quad s.t\ By \le r - Ax \\
\qquad y \ge 0\ .
\end{cases}$$

$$\Rightarrow\begin{cases}
\max\limits_{y\in\Omega(x)} f(x,y) = c^Tx + x^TQ_2x + (d + 2Q_1x)^Ty + y^TQ_0y \\
\quad s.t\ By \le r - Ax \\
\qquad y \ge 0.
\end{cases}$$

Bearing in mind that $c^Tx + x^TQ_2x$ is a constant term, it can be assumed that $c = Q_2 = 0$ to ignore the term $c^Tx + x^TQ_2x$ and then without loss of generality the lower level programming problem can be solved. Thus the optimal solution to the lower level (follower) can be obtained by solving the following problem:

$$(P_x): \begin{cases}
\max\limits_{y\in\Omega(x)} f(x,y) = (d + 2Q_1x)^Ty + y^TQ_0y \\
\quad s.t\ By \le r - Ax \\
\qquad y \ge 0.
\end{cases}$$

It is assumed that Q_0 is negative definite so that there will be a unique optimal solution to the second level problem. That is to say, $R(x)$ is a singleton for all $x \in \Omega$ and the LFQBLPP is well posed. As a consequence, the LFQBLPP is equivalent to the following BLPP, which will be considered in the sequel:

$$\begin{cases} \max\limits_{x\in\Omega} F(x,y) = \frac{a_1+a_2{}^T x+a_3{}^T y}{b_1+b_2{}^T x+b_3{}^T y} \\ \qquad\qquad where\ y\ solves \\ \max\limits_{y\in\Omega(x)} f(x,y) = (d+2Q_1x)^T y + y^T Q_0 y \\ \qquad subject\ to\ \ Ax+By \le r \\ \qquad\qquad\qquad y \ge 0. \end{cases} \tag{5.2}$$

The conventional solution approach to the BLPP is to transform the original two-level problems into a single level one by replacing the lower level optimization problem with its KKT necessary and sufficient conditions. Now consider the lower level problem, for a given x

$$\begin{cases} \max\limits_{y\in\Omega(x)} \ \ (d+2Q_1x)^T y + y^T Q_0 y \\ \text{subject to } By \le r - Ax \\ \qquad\qquad y \ge 0, \end{cases}$$

Here, $f(x,y) = (d+2Q_1x)^T y + y^T Q_0 y$,

$$g(x,y) = Ax + By - r \le 0 \Rightarrow g(x,y) = r - Ax - By \ge 0.$$

Define the Lagrangian function $L(x,y,\lambda)$ as $L(x,y,\lambda) = f(x,y) + \lambda^T g(x,y)$, where $\lambda \ge 0$ is the vector of Lagrangian multipliers. Applying the KKT conditions yields,

$$\frac{\partial L}{\partial y} \le 0 \Rightarrow d + 2Q_1x + 2Q_0y - B^T\lambda \le 0, \tag{5.3}$$

$$\frac{\partial L}{\partial \lambda} \ge 0 \Rightarrow g(x,y) \ge 0 \Rightarrow r - Ax - By \ge 0, \tag{5.4}$$

$$y^T \frac{\partial L}{\partial y} = 0 \Rightarrow y^T(d + 2Q_1x + 2Q_0y + B^T\lambda) = 0, \tag{5.5}$$

$$\lambda^T \frac{\partial L}{\partial \lambda} = 0 \Rightarrow \lambda^T(r - Ax - By) = 0. \tag{5.6}$$

In equation (5.3), introducing the slack variable and converting into equality form give

$$d + 2Q_1x + 2Q_0y - B^T\lambda + Iu = 0 \ \text{ Or}$$

$$-2Q_1x - 2Q_0y + B^T\lambda - Iu = d, u \ge 0. \tag{5.7}$$

Using equations (5.5) and (5.7) produce

$$y^T u = 0. \tag{5.8}$$

In equation (5.4), introducing the surplus variable, we get

$$r - Ax - By - Iv = 0.$$

Or $Ax + By + Iv = r, v \geq 0.$ (5.9)

From equations (5.6) and (5.9) we get:

$$\lambda^T v = 0. \tag{5.10}$$

Equations (5.7), (5.8), (5.9) and (5.10) give the KKT conditions corresponding to the lower level objective function. Because of the BLPP with the concave lower level problem is equivalent to the single level programming problem by replacing the lower level problem with KKT necessary and sufficient conditions. Thus, the given LFQBLP problem becomes an LFPPCC given by:

$$\begin{cases} \max\limits_{x\in\Omega} F(x,y) = \frac{a_1 + a_2{}^T x + a_3{}^T y}{b_1 + b_2{}^T x + b_3{}^T y} \\ subject\ to \quad Ax + By + Iv = r \\ \qquad\qquad -2Q_1 x - 2Q_0 y + B^T \lambda - Iu = d \\ \qquad\qquad \lambda^T v = 0 \\ \qquad\qquad y^T u = 0 \\ \qquad\qquad x, y, \lambda, u, v \geq 0. \end{cases} \tag{5.11}$$

Here, I is the identity matrix of the appropriate dimension and the constraint $\lambda^T v = 0$ and $y^T u = 0$ represents the complementary constraint. As the objective function is linear fractional, it is both pseudoconcave and pseudoconvex, and thus its optimal solution will be at an extreme point. So that finding this extreme point which satisfies the condition $\lambda^T v = 0$ and $y^T u = 0$ is the interest of the present work.

Remark 1 (Wang *et al.*, 2007): If $f(x,y)$ is continuous, concave and constraint qualification holds for the problem (p_x) with fixed x at x^* then by applying a KKT necessary and sufficient conditions, that is (x^*, y^*) solves the problem (5.1), then there exist $u^* \in \mathbb{R}^m$, $\lambda^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^{n_1} \geq 0$, such that $(x^*, y^*, u^*, \lambda^*, v^*)$ is the optimal solution of the problem (5.11). Hence, we recall that a point (x^*, y^*) is called an optimal solution of the problem (5.1) if (x^*, y^*) is an optimal solution of the problem (5.11).

Similarly for the minimization case the LFQBLPP can be written as follows:

$$\begin{cases} \min\limits_{x\in\Omega} F(x,y) = \frac{a_1 + a_2{}^T x + a_3{}^T y}{b_1 + b_2{}^T x + b_3{}^T y} \\ \qquad where\ for\ a\ given\ y\ solves \\ \min\limits_{y\in\Omega(x)} f(x,y) = c^T x + d^T y + (x^T, y^T) Q (x^T, y^T)^T \\ \quad subject\ to \quad (x,y) \in \Omega, \end{cases} \tag{5.12}$$

where $\Omega = \{(x,y) \in \mathbb{R}^{n_1+n_2}: Ax + By \leq r\,; x, y \geq 0\}$; $a_1, b_1 \in \mathbb{R}$; $a_2, b_2, c \in \mathbb{R}^{n_1}$; $a_3, b_3, d \in \mathbb{R}^{n_2}$; $A \in \mathbb{R}^{m\times n_1}$; $B \in \mathbb{R}^{m\times n_2}$; $r \in \mathbb{R}^m$; $x \in X \subseteq \mathbb{R}^{n_1}$; $y \in Y \subseteq \mathbb{R}^{n_2}$ and Q is an $((n_1 + n_2) \times (n_1 + n_2))$ dimensional symmetric positive semi-definite matrix (*i.e.*, $Q \in R^{n_1+n_2} \times R^{n_1+n_2}$). With

$$Q = \begin{bmatrix} Q_2 & {Q_1}^T \\ Q_1 & Q_0 \end{bmatrix},$$

where $Q_0 \in \mathbb{R}^{n_2\times n_2}, Q_1 \in \mathbb{R}^{n_2\times n_1}\ and\ Q_2 \in \mathbb{R}^{n_1\times n_1}$ are matrices of conformal dimensions. In order to ensure that the problem (5.12) is well-posed, we assume that the common constraint region $\Omega = \{(x,y): Ax + By \leq r, x, y \geq 0\}$ is non-empty and bounded. In addition, it is also assumed that $b_1 + {b_2}^T x + {b_3}^T y > 0, \forall(x,y) \in \Omega$. If this is not so, it suffices to consider the linear fractional objective function as $\frac{-(a_1+{a_2}^T x+{a_3}^T y)}{-(b_1+{b_2}^T x+{b_3}^T y)}$. Define, $\Omega(x) = \{y \in \mathbb{R}^{n_2}\,; (x,y) \in \Omega\}$ gives the feasible region of the lower level problem, for any arbitrary value of x. Using the same process as above the LFQBLPP is equivalent to the following BLPP, which will be considered in the sequel:

$$\begin{cases} \min\limits_{x\in\Omega} F(x,y) = \frac{a_1+{a_2}^T x+{a_3}^T y}{b_1+{b_2}^T x+{b_3}^T y} \\ \qquad\qquad where\ y\ solves \\ \min\limits_{y\in\Omega(x)} f(x,y) = (d + 2Q_1 x)^T y + y^T Q_0 y \\ \qquad subject\ to\ \ Ax + By \leq r \\ \qquad\qquad\quad y \geq 0. \end{cases} \tag{5.13}$$

The conventional solution approach to the BLPP is to transform the original two-level problems into a single level one by replacing the lower level optimization problem with its KKT necessary and sufficient conditions.

With this now consider the lower level problem, for a given x

$$(q_x): \begin{cases} \min\limits_{y\in\Omega(x)} \quad (d + 2Q_1 x)^T y + y^T Q_0 y \\ \text{subject to } By \leq r - Ax \\ \qquad\qquad y \geq 0, \end{cases}$$

Here, $f(x,y) = (d + 2Q_1 x)^T y + y^T Q_0 y,\ \ g(x,y) = Ax + By - r \leq 0.$

Define the Lagrangian function $L(x,y,\lambda)$ as $L(x,y,\lambda) = f(x,y) + \lambda^T g(x,y)$, where $\lambda \geq 0$ is the vector of Lagrangian multipliers.

Applying the KKT conditions, we have that

$$\frac{\partial L}{\partial y} \geq 0 \Rightarrow d + 2Q_1 x + 2Q_0 y + B^T\lambda \geq 0, \tag{5.14}$$

$$\frac{\partial L}{\partial \lambda} \leq 0 \Rightarrow g(x,y) \leq 0 \Rightarrow Ax + By - r \leq 0, \tag{5.15}$$

$$y^T \frac{\partial L}{\partial y} = 0 \Rightarrow y^T(d + 2Q_1x + 2Q_0y + B^T\lambda) = 0, \tag{5.16}$$

$$\lambda^T \frac{\partial L}{\partial \lambda} = 0 \Rightarrow \lambda^T(Ax + By - r) = 0, \tag{5.17}$$

In equation (5.14), introducing the surplus variable and converting into equality form, we get:

$d + 2Q_1x + 2Q_0y + B^T\lambda - Iu = 0$ Or

$$-2Q_1x - 2Q_0y - B^T\lambda + Iu = d, u \geq 0. \tag{5.18}$$

Using equations (5.16) and (5.18) produce

$$y^Tu = 0. \tag{5.19}$$

In equation (5.15), introducing the slack variable, we get:

$Ax + By - r + Iv = 0$ Or

$$Ax + By + Iv = r, v \geq 0. \tag{5.20}$$

from equations (5.17) and (5.20), we get

$$\lambda^Tv = 0. \tag{5.21}$$

Equations (5.18), (5.19), (5.20) and (5.21) gives the KKT conditions corresponding to the lower level objective function. Because of the BLPP with the convex lower level problem is equivalent to the single level programming problem by replacing the lower level problem with KKT necessary and sufficient conditions. Thus, the given LFQBLPP becomes an LFPPCC and it is given as follows:

$$\begin{cases} \min\limits_{x\in\Omega} F(x,y) = \frac{a_1 + {a_2}^T x + {a_3}^T y}{b_1 + {b_2}^T x + {b_3}^T y} \\ subject\ to \quad Ax + By + Iv = r \\ \qquad -2Q_1x - 2Q_0y - B^T\lambda + Iu = d \\ \qquad \lambda^Tv = 0 \\ \qquad y^Tu = 0 \\ \qquad x, y, \lambda, u, v \geq 0. \end{cases} \tag{5.22}$$

Here, I is the identity matrix of the appropriate dimension and the constraint $\lambda^Tv = 0$ and $y^Tu = 0$ represents the complementary constraint. As the objective function is linear fractional, therefore, it is both pseudoconcave and pseudoconvex, and thus its optimal solution will be at an extreme point. As the objective function in linear fractional problem is both pseudoconvex and pseudoconcave, hence it is pseudo linear. So every pseudo linear function is strictly quasi linear thus every local optima of pseudo linear function is also global optima. We are interested in finding that extreme point which satisfies the condition $\lambda^Tv = 0$ and $y^Tu = 0$.

Remark 2 (Wang *et al*., 2007): If $f(x,y)$ is continuous, convex and constraint qualification holds for the problem (q_x) with fixed x at x^* then by applying a KKT necessary and sufficient conditions, that is (x^*,y^*) solves the problem (5.12), then there exist $u^* \in \mathbb{R}^m$, $\lambda^* \in \mathbb{R}^m$ and $v^* \in \mathbb{R}^{n_1} \geq 0$, such that $(x^*,y^*,u^*,\lambda^*,v^*)$ is the optimal solution of the problem (5.22). Hence, we recall that a point (x^*,y^*) is called an optimal solution of the problem (5.12) if (x^*,y^*) is an optimal solution of the problem (5.22).

Regularity Condition

In the Introduction we have seen several possible reformulations of the bilevel programming problem into ordinary one-level problems. One uses the Karush-Kuhn-Tucker conditions of the lower level problem to replace them by:

$$\begin{cases} \max\limits_{x\in\Omega} F(x,y) = \frac{a_1+{a_2}^T x + {a_3}^T y}{b_1+{b_2}^T x+{b_3}^T y} \\ subject\ to \quad Ax+By+Iv=r \\ \qquad\qquad -2Q_1x-2Q_0y+B^T\lambda-Iu=d \\ \qquad\qquad \lambda^T v=0 \\ \qquad\qquad y^T u=0 \\ \qquad\qquad x,y,\lambda,u,v \geq 0. \end{cases} \tag{5.11}$$

Note that (5.1) and (5.11) are equivalent provided that the lower level programming problem (5.1) is a convex parametric optimization problem satisfying Mangasarian-Fromowitz constraint qualifications at all feasible points $y \in \Omega(x)$, $g(x,y) \leq 0$ and we settle upon the computation of global optimistic optimal solutions. Without convexity assumption, problem (5.11) has a larger feasible set including not only global optimal solutions of the lower level problem but also all local optimal solutions and also all stationary points. Hence, the optimal function value of (5.11) is never larger than that of the bilevel programming problem (5.1). Moreover, problem (5.11) is a smooth optimization problem which could be used as an indication for an easier treatment. But this is not completely correct since at least the regularity assumptions which are needed for successfully handling smooth optimization problems are never satisfied. To find a better approach, a non-smooth equivalent of the bilevel programming problem (5.1) is formulated below. Then, respective regularity assumptions for non-smooth optimization problems can be satisfied.

The main difficulty concerning the reformulation (5.11) is the violation of most of the usual constraint qualifications. Undoubtedly among the better known Constraint Qualifications are the linear independence and Mangasarian-Fromowitz constraint qualifications.

Definition 2.1 Let z^* be a feasible point of the lower level problem of (5.1). We then say that

(a) The linear independence constraint qualification holds at z^* if the gradient vectors $\nabla g_i(z^*), \forall\, i \in I_g$ are linearly independent, where $I_g = \{i | g_i(z^*) = 0$ is the set of the active inequalities of g in z^*;

(b) The Mangasarian-Fromowitz constraint qualification, holds at z^* if the gradient vectors $\nabla g_i(z^*), \forall\, i \in I_g$ are linearly independent and there exists a vector $d \in \mathbb{R}^n$ such that

$$\nabla g_i(z^*)^T d < 0, \qquad \forall\, i \in I_g$$

Theorem 1 (Scheel and Scholtes, 1998): If the Karush-Kuhn-Tucker conditions of the lower level problem of (5.1) are the part of the constraints of an optimization problem, then the Mangasarian-Fromowitz constraint qualification is violated at every feasible point.

According to the above two remarks, we can solve the problem (5.11) and (5.22) to obtain the optimum of the problem (5.1) and (5.12) respectively. So based on the reformation, we concerned a modified simplex algorithm is a numerical algorithm compatible for the optimization problem we proposed the modified simplex algorithms to solve the LFQBLPP by the use of the favorable characteristic of the modified simplex algorithms. It is obvious that the complementary conditions are the difficulties for solving the problem (5.11) and (5.22). So we get rid of the complementary slack conditions to simplify the problem (5.11) and (5.22).

5.2. The Modified Simplex Method

The modified simplex method exploits the key fact that with the exception of the complementarity constraint, the KKT conditions in the convenient form obtained above are nothing more than linear programming constraints. Furthermore, the complementarity constraint simply implies that it is not permissible for both complementary variables of any pair to be non-degenerate basic variables (the only variables that are greater than zero) when non-degenerate basic feasible solutions are considered. Therefore, the problem reduces to finding an initial basic feasible solution to any linear programming problem that has these constraints, subject to this additional restriction on the identity of the basic variables. (This initial basic feasible solution may be the only feasible solution in this case).

To find a basic feasible solution for the problem; apply the simplex method (with one modification) to the following LFPP.

$$\underset{x\in\Omega}{max/min}\, F(x,y) = \frac{a_1+{a_2}^T x+{a_3}^T y}{b_1+{b_2}^T x+{b_3}^T y},$$

Subject to the linear constraints with complementary conditions obtained from the KKT conditions. The one modification in the simplex method is the following change in the procedure for selecting an entering basic variable.

Restricted-Entry Rule: When you are choosing an entering basic variable, exclude any non-basic variable from consideration whose complementary variable is already a basic variable; the choice should be made from the other non-basic variables according to the usual criterion for the simplex method.

5.2.1. The Development of Modified Simplex Method

The simplex method has been developed by Dantzig in (1947). It provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner such that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever be repeated and there is no need to go back to the vertex which is already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number throughout the steps.

The Simplex algorithm is an iterative procedure for solving linear programming problems. It consists of:

(i) Having a trail basic feasible solution to constraint equations.

(ii) Testing whether an optimal solution is reached or not.

(iii) Improving the first trial solution by a set of rules, and repeating the process until an optimal solution is obtained.

The computational procedure requires at most m (equal to the number of equations) non-zero variables in the solution at any step. In case of less than m non-zero variables at any stage of computations, the degeneracy arises in linear programming problem.

Further, it is being instructive to note that a feasible solution at any iteration is related to the feasible solution of the successive iteration in the following way. One of the non-basic variables (which is zero) at the present iteration becomes basis (nonzero) at the upcoming iteration, and is called an entering variable. Then to compensate the newly converted nonzero variable, one of the basic variables (which is non-zero) at the present iteration becomes non-basic (zero) at the upcoming iteration and called a departing variable. The other non-basic variables remain zero, and basic variables remain non-zero (though their values may change).

5.2.2. Modified Simplex Method for Solving Linear Fractional/Quadratic Bi-Level Programming Problem

This section deals with the solution of the LFQBLPP by using an MSM. This method can be successfully adapted to high computational speed. It can be applied if the constraints of the problem are linear function. *i.e.* for maximization case our problem is of the form:

$$\begin{cases} \max\limits_{x\in\Omega} F(x,y) = \frac{a_1+{a_2}^T x+{a_3}^T y}{b_1+{b_2}^T x+{b_3}^T y} \\ \qquad\qquad\qquad where\ y\ solves \\ \max\limits_{y\in\Omega(x)} f(x,y) = c^T x + d^T y + (x^T, y^T)Q(x^T, y^T)^T \\ \qquad\qquad subject\ to \qquad (x,y)\in\Omega, \end{cases}$$

where $\Omega = \{(x,y)\in\mathbb{R}^{n_1+n_2}: Ax+By\le r\,; x,y\ge 0\}$; $a_1, b_1\in\mathbb{R}$; $a_2, b_2, c\in\mathbb{R}^{n_1}$; $a_3, b_3, d\in\mathbb{R}^{n_2}$; $A\in\mathbb{R}^{m\times n_1}$; $B\in\mathbb{R}^{m\times n_2}$; $r\in\mathbb{R}^m$; $x\in X\subseteq\mathbb{R}^{n_1}$; $y\in Y\subseteq\mathbb{R}^{n_2}$ and Q is an $((n_1+n_2)\times(n_1+n_2))$ dimensional symmetric negative semi-definite matrix and $F(x,y)$ and $f(x,y)$ are the objective functions of the leader and concave objective function of the follower, respectively. By using the KKT necessary and sufficient optimality condition we can reformulate the above LFQBLPP into an LFPPCC. So the LFPPCC can be defined as follows:

$$\begin{cases} \max\limits_{x\in\Omega} F(x,y) = \frac{a_1+{a_2}^T x+{a_3}^T y}{b_1+{b_2}^T x+{b_3}^T y} \\ subject\ to \quad Ax+By+Iv=r \\ \qquad\qquad -2Q_1x-2Q_0y+B^T\lambda-Iu=d \text{ And} \\ \qquad\qquad \lambda^T v=0 \\ \qquad\qquad y^T u=0 \\ \qquad\qquad x,y,\lambda,u,v\ge 0. \end{cases} \tag{5.23}$$

For simplicity the above problem without the complementary conditions can be re-written as follows:

$$\begin{cases} \max\limits_{x \in \Omega} F(z) = \frac{az+a_1}{bz+b_1} \\ subject\ to \\ \qquad \begin{bmatrix} A & B & I & 0 & 0 \\ -2Q_1 & -2Q_0 & 0 & B^T & -I \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ \lambda \\ u \end{bmatrix} = \begin{bmatrix} r \\ d \end{bmatrix} \\ \qquad\qquad x, y, v, \lambda, u \geq 0, \end{cases}$$

where $a = (a_2, a_3) \in \mathbb{R}^{n_1+n_2}$; $b = (b_2, b_3) \in \mathbb{R}^{n_1+n_2}$; $z = (x, y) \in \mathbb{R}^{n_1+n_2}$; $a_1, b_1 \in \mathbb{R}$; $r \in \mathbb{R}^m$; $d \in \mathbb{R}^{n_2}$.

The above problem becomes

$$\begin{cases} \max\limits_{x \in \Omega} F(z) = \frac{az+a_1}{bz+b_1} \\ subject\ to \\ \qquad U\mathrm{X} = V \\ \qquad \mathrm{X} \geq 0, \end{cases}$$

where $U = \begin{bmatrix} A & B & I & 0 & 0 \\ -2Q_1 & -2Q_0 & 0 & -B^T & I \end{bmatrix}$; $\mathrm{X} = \begin{bmatrix} x \\ y \\ v \\ \lambda \\ u \end{bmatrix}$; $V = \begin{bmatrix} r \\ d \end{bmatrix}$.

Here, $U \in \mathbb{R}^{(m+n_2)\times(n_1+2n_2+2m)}$, $\mathrm{X} \in \mathbb{R}^{n_1+2n_2+2m}$, $V \in \mathbb{R}^{m+n_2}$.

For minimization case our problem is of the form:

$$\begin{cases} \min\limits_{x \in \Omega} F(x, y) = \frac{a_1 + a_2{}^T x + a_3{}^T y}{b_1 + b_2{}^T x + b_3{}^T y} \\ \qquad\qquad where\ y\ solves \\ \min\limits_{y \in \Omega(x)} f(x, y) = c^T x + d^T y + (x^T, y^T) Q (x^T, y^T)^T \\ \qquad subject\ to \quad (x, y) \in \Omega, \end{cases}$$

where $\Omega = \{(x, y) \in \mathbb{R}^{n_1+n_2} \colon Ax + By \leq r\ ; x, y \geq 0\}$; $a_1, b_1 \in \mathbb{R}$; $a_2, b_2, c \in \mathbb{R}^{n_1}$; $a_3, b_3, d \in \mathbb{R}^{n_2}$; $A \in \mathbb{R}^{m \times n_1}$; $B \in \mathbb{R}^{m \times n_2}$; $r \in \mathbb{R}^m$; $x \in X \subseteq \mathbb{R}^{n_1}$; $y \in Y \subseteq \mathbb{R}^{n_2}$ and Q is an $((n_1 + n_2) \times (n_1 + n_2))$ dimensional symmetric positive semi-definite matrix and $F(x, y)$ and $f(x, y)$ are the objective functions of the leader and convex objective function of the follower, respectively. By using the KKT necessary and sufficient optimality condition we can reformulate the above LFQBLPP into an LFPPCC. So the LFPPCC can be defined as follows:

$$\begin{cases} \min\limits_{x\in\Omega} F(x,y) = \frac{a_1 + {a_2}^T x + {a_3}^T y}{b_1 + {b_2}^T x + {b_3}^T y} \\ subject\ to \quad Ax + By + Iv = r \\ \qquad\qquad -2Q_1 x - 2Q_0 y - B^T\lambda + Iu = d \\ \qquad\qquad \lambda^T v = 0 \\ \qquad\qquad y^T u = 0 \\ \qquad\qquad x, y, \lambda, u, v \geq 0. \end{cases} \tag{5.24}$$

Similarly, the above problem without the complementary constraint can be re-written as follows:

$$\begin{cases} \min\limits_{x\in\Omega} F(z) = \frac{az + a_1}{bz + b_1} \\ subject\ to \\ \qquad \begin{bmatrix} A & B & I & 0 & 0 \\ -2Q_1 & -2Q_0 & 0 & -B^T & I \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ \lambda \\ u \end{bmatrix} = \begin{bmatrix} r \\ d \end{bmatrix} \\ \qquad x, y, v, \lambda, u \geq 0, \end{cases}$$

where $a = (a_2, a_3) \in \mathbb{R}^{n_1+n_2}$; $b = (b_2, b_3) \in \mathbb{R}^{n_1+n_2}$; $z = (x, y) \in \mathbb{R}^{n_1+n_2}$; $a_1, b_1 \in \mathbb{R}$; $r \in \mathbb{R}^m$; $d \in \mathbb{R}^{n_2}$.

The above problem becomes

$$\begin{cases} \min\limits_{x\in\Omega} F(z) = \frac{az + a_1}{bz + b_1} \\ subject\ to \\ \qquad U\mathrm{X} = V \\ \qquad \mathrm{X} \geq 0, \end{cases}$$

where $U = \begin{bmatrix} A & B & I & 0 & 0 \\ -2Q_1 & -2Q_0 & 0 & -B^T & I \end{bmatrix}$; $\mathrm{X} = \begin{bmatrix} x \\ y \\ v \\ \lambda \\ u \end{bmatrix}$; $V = \begin{bmatrix} r \\ d \end{bmatrix}$.

Here, $\mathrm{X} \in \mathbb{R}^{n_1+2n_2+2m}$, $U \in \mathbb{R}^{(m+n_2)\times(n_1+2n_2+2m)}$, $V \in \mathbb{R}^{m+n_2}$.

Suppose that an extreme point of the feasible region with basis B is given such that $\mathrm{X}_B = B^{-1}V > 0$ and $\mathrm{X}_N = 0$. Where X_B is the basic vector and X_N is the non-basic vector. B is $(m + n_2) \times (m + n_2)$ invertible matrix. Since the current point is an extreme point with $\mathrm{X}_N = 0$, the non-basic variable cannot be decreased further as it would violate the non-negativity restriction. Let

$$\begin{aligned} r^T = ({r_B}^T, {r_N}^T) &= \nabla F(z)^T - \nabla_B F(z)^T B^{-1} U \\ &= [(\nabla_B F(z)^T, \nabla_N F(z)^T) - \nabla_B F(z)^T B^{-1}(B, N)] \end{aligned}$$

$$= [\nabla_B F(z)^T - \nabla_B F(z)^T B^{-1} B,\ \ \nabla_N F(z)^T - \nabla_B F(z)^T B^{-1} N]$$
$$= [0, \nabla_N F(z)^T - \nabla_B F(z)^T B^{-1} N].$$

Thus using an MSM developed to solve the numerical example to apply the simplex process, first we find $\nabla F(z)$ by calculating the partial derivatives of the linear fractional objective function to find the entering variable. In this approach we define the formula to find ${r_N}^T$ from $\nabla_N F(z)^T, \nabla_B F(z)^T, B^{-1}$ and N as follows: ${r_N}^T = \nabla_N F(z)^T - \nabla_B F(z)^T B^{-1} N$, where r_N denotes the non-basic components of the reduced gradient vector r^T.

Criterion 1: (Choice of the Entering Variable)

To identify the entering variable, Find $r_s = \max\{r_i : r_i \geq 0\}$ for the case of maximization and Find $r_s = \min\{r_i : r_i \leq 0\}$ for the case of minimization, where r_i is the i^{th} component of r_N. The non-basic variable X_j is increased, and the basic variables are modified to maintain feasibility.

Criterion 2: (Choice of the Leaving (departing) variable)

To determine the basic variable which is going to be leaved, the following minimum ratio test can be applied.

$$\mathrm{X}_{B_r} = \frac{\mathrm{V}_{B_r}}{\mathrm{Y}_{r_j}} = \min_{1 \leq i \leq m+n_2} \left\{ \frac{\mathrm{V}_{B_i}}{\mathrm{Y}_{i_j}} : \mathrm{Y}_{i_j} > 0 \right\}, \text{ where } \mathrm{Y}_{i_j} = B^{-1} u_j,\ u_j \text{ is the } j^{th} \text{ column of } U.$$

Replace the variable X_B, by the variable X_j. Update the table correspondingly by pivoting at Y_{rj}.

Remark 3 (Calvete and Galé, 2004): Notice that, for any fixed $(x, y) \in IR$, y is an optimal solution to (P_x) and (q_x). Hence, by applying KKT necessary and sufficient conditions, there exists $\lambda \in \mathbb{R}^m$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^{n_1}$ such that (x, y, λ, u, v)satisfies

$$\begin{cases} Ax + By + Iv = r \\ -2Q_1 x - 2Q_0 y - B^T \lambda + Iu = d \\ \lambda^T v = 0 \\ y^T u = 0 \\ \quad x, y, \lambda, u, v \geq 0, \end{cases} \qquad (*)$$

similarly if (x, y, λ, u, v) satisfies $(*)$ then $(x, y) \in IR$.

5.4. Optimality Conditions for the Linear Fractional/Quadratic Bi-Level Programming Problem

In this section, necessary and sufficient optimality conditions for a pair (x, y) to be an optimal solution of LFQBLPP are driven.

Theorem 2 (Calvete and Galé, 2004): (x^*, y^*) is an optimal solution to the LFQBLPP if and only if there exist $\lambda^* \in \mathbb{R}^m$, $u^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^{n_1}$ such that $(x^*, y^*, \lambda^*, u^*, v^*)$ is an optimal solution to the following one level (NLP). That is,

$$\text{NLP:}\begin{cases} \min\limits_{(x,y,\lambda,u,v)} F(x,y) = \frac{a_1+a_2{}^T x+a_3{}^T y}{b_1+b_2{}^T x+b_3{}^T y} \\ subject\ to \begin{cases} Ax + By + Iv = r \\ -2Q_1 x - 2Q_0 y - B^T\lambda + Iu = d \\ \lambda^T v = 0 \\ y^T u = 0 \\ x, y, \lambda, u, v \geq 0. \end{cases} \end{cases}$$

Proof: (⇒) let (x^*, y^*) is an optimal solution to the LFQBLPP. Taking into account remark 3, since $(x^*, y^*) \in IR$, there exist $\lambda^* \in \mathbb{R}^m$, $u^* \in \mathbb{R}^m$, $v^* \in \mathbb{R}^{n_1}$ such that $(x^*, y^*, \lambda^*, u^*, v^*)$ satisfies all the constraint in the above problem NLP, that is to say it is a feasible solution to the problem NLP. If $(x^*, y^*, \lambda^*, u^*, v^*)$ was not an optimal solution to NLP, there would exist $(\hat{x}, \hat{y}, \hat{\lambda}, \hat{u}, \hat{v})$ satisfying all the constraint in the above NLP problem, thus $(x^*, y^*) \in IR$, such that

$$\frac{a_1+a_2{}^T\hat{x}+a_3{}^T\hat{y}}{b_1+b_2{}^T\hat{x}+b_3{}^T\hat{y}} < \frac{a_1+a_2{}^T x^*+a_3{}^T y^*}{b_1+b_2{}^T x^*+b_3{}^T y^*}.$$

This, together with the fact that $(\hat{x}, \hat{y}) \in IR$, contradicts the optimality of (x^*, y^*). Therefore, $(x^*, y^*, \lambda^*, u^*, v^*)$ solves the problem NLP.

Conversely, (⇐) let $(x^*, y^*, \lambda^*, u^*, v^*)$be an optimal solution to the problem NLP. Since, $(x^*, y^*, \lambda^*, u^*, v^*)$ satisfies all the above constraints, we conclude that $(x^*, y^*) \in IR$. On the other hand, for any fixed $(x, y) \in IR$, there exist $\lambda \in \mathbb{R}^m, u \in \mathbb{R}^m$ and $v \in \mathbb{R}^{n_1}$ such that (x, y, λ, u, v) is a feasible solution to the problem NLP. Moreover, since $(x^*, y^*, \lambda^*, u^*, v^*)$is an optimal solution to the problem NLP, $F(x^*, y^*) \leq F(x, y)$. Therefore, (x^*, y^*) solves the LFQBLPP. This complete the proof. And we use the same procedure for maximization case.

In this project, the optimal solution is obtained when all $r_N \leq 0$ for maximization case (the optimal solution is obtained when all $r_N \geq 0$ for minimization case) resulting in a KKT point.

The optimal solution of LFPP will maximize (minimize) the objective function if and only if $\lambda^T v = 0$ and $y^T u = 0$. Also if $\lambda^T v \neq 0$ or $y^T u \neq 0$ find the next solution to the above problem with the restricted basis entry rule, until the condition $\lambda^T v = 0$ and $y^T u = 0$.

5.4.1. Convergence

Here, it is assumed that $\mathrm{X}_B > 0$ for each extreme point. The above method moves from one extreme point to another extreme point. By the non-degeneracy assumption, the objective function strictly increases for the maximization case and also (the objective function strictly decreases for minimization case) at each iteration so that the generated extreme points are distinct. There is only a finite number of these points and hence, the procedure stops in a finite number of steps.

Algorithm for solving maximization case of LFQBLPP (Arora and Arora, 2011)

Step 1: Consider the LFQBLPP for a given value of x, take the lower level programming problem in equation (5.2).

$f(x, y) = (d + 2Q_1 x)^T y + y^T Q_0 y$ and $g(x, y) = r - Ax - By \geq 0$.

Define the Lagrangian function $L(x, y, \lambda) = f(x, y) + \lambda^T g(x, y)$. Then apply the KKT conditions and convert LFQBLPP to LFPP with the condition that $\lambda^T v = 0$ and $y^T u = 0$.

Step 2: Remove the condition $\lambda^T v = 0$ and $y^T u = 0$ and solve the LFPP. To solve the LFPP, find an initial basic feasible solution.

Step 3: Compute the vector ${r_N}^T = \nabla_N F(\mathrm{X}_s)^T - \nabla_B F(\mathrm{X}_s)^T B^{-1} N$. If $r_N \leq 0$, the current point X_s is an optimal solution of LFPP. Go to step 6. Otherwise, go to step 4.

Step 4: Let $r_s = \max\{r_i : r_i \geq 0\}$, where r_i is the i^{th} component of $\boldsymbol{r_N}$. Determine the basic variable X_{B_r} to leave the basis by the minimum ratio test defined as

$$\frac{\mathrm{V}_{B_r}}{\mathrm{Y}_{r_j}} = \min_{1 \leq i \leq m+n_2} \left\{ \frac{\mathrm{V}_{B_i}}{\mathrm{Y}_{ij}} : \mathrm{Y}_{ij} > 0 \right\},$$

where $\mathrm{Y}_j = B^{-1} u_j$; u_j is the j^{th} column of U. Go to step 5.

Step 5: Replace the variable X_{B_r} by the variable X_j. Update the table correspondingly by pivoting at Y_{r_j}. Let the current solution be X_{s+1}. Replace s by $s + 1$. Go to step 3.

Step 6: (a) Check $\lambda^T v = 0$ and $y^T u = 0$. If $\lambda^T v = 0$ and $y^T u = 0$, then this solution will be the optimal solution of LFQBLPP.

(b) If at least one of $\lambda^T v \neq 0$ or $y^T u \neq 0$, find such a solution with restricted basis entry, such that the condition $\lambda^T v = 0$ and $y^T u = 0$ is satisfied. It will be the optimal solution of LFQBLPP.

Algorithm for solving minimization case of LFQBLPP (Arora and Arora, 2011)

Step 1: Consider the LFQBLPP for a given value of x, take the lower level programming problem in equation (5.13).

$f(x, y) = (d + 2Q_1 x)^T y + y^T Q_0 y$ and $g(x, y) = Ax + By - r \leq 0$.

Define the Lagrangian function $L(x, y, \lambda) = f(x, y) + \lambda^T g(x, y)$. Then apply the KKT conditions and convert LFQBLPP to LFPP with the condition that $\lambda^T v = 0$ and $y^T u = 0$.

Step 2: Remove the condition $\lambda^T v = 0$ and $y^T u = 0$ and solve the LFPP. To solve the LFPP, find an initial basic feasible solution.

Step 3: Compute the vector ${r_N}^T = \nabla_N F(\mathrm{X}_s)^T - \nabla_B F(\mathrm{X}_s)^T B^{-1} N$. If $r_N \geq 0$, the current point X_s is an optimal solution of LFPP. Go to step 6. Otherwise, go to step 4.

Step 4: Let $r_s = \min\{r_i : r_i \leq 0\}$, where r_i is the i^{th} component of $\boldsymbol{r_N}$. Determine the basic variable X_{B_r} to leave the basis by the minimum ratio test defined as

$$\frac{\mathrm{V}_{B_r}}{\mathrm{Y}_{r_j}} = \min_{1 \leq i \leq m+n_2} \left\{\frac{\mathrm{V}_{B_i}}{\mathrm{Y}_{ij}} : Y_{ij} > 0\right\},$$

where $\mathrm{Y}_j = B^{-1} u_j$; u_j is the j^{th} column of U. go to step 5.

Step 5: Replace the variable X_{B_r} by the variable X_j. Update the table correspondingly by pivoting at Y_{r_j}. Let the current solution be X_{s+1}. Replace s by $s + 1$. Go to step 3.

Step 6: (a) Check $\lambda^T v = 0$ and $y^T u = 0$. If $\lambda^T v = 0$ and $y^T u = 0$, then this solution will be the optimal solution of LFQBLPP.

(b) If at least one of $\lambda^T v \neq 0$ or $y^T u \neq 0$, find such a solution with restricted basis entry rule, such that the condition $\lambda^T v = 0$ and $y^T u = 0$ is satisfied. It will be the optimal solution of LFQBLPP.

5.5. Computational Results

Example 5.1 (Malhotra and Arora, 2001): Consider the following LFQBLPP.

$$\begin{cases} \max\limits_{x_1} F(x_1,x_2) = \frac{8x_1+3x_2}{5x_1+2x_2+1} \\ \qquad\qquad where\ x_2\ solves \\ \max\limits_{x_2} f(x_1,x_2) = 2x_1 - 3x_2 - {x_1}^2 - {x_2}^2 \\ subject\ to \quad x_1 + 4x_2 \le 4 \\ \qquad\qquad\quad x_1 + x_2 \le 2 \\ \qquad\qquad\quad x_1, x_2 \ge 0. \end{cases}$$

Solution:

From the follower function $f(x_1,x_2) = 2x_1 - 3x_2 - {x_1}^2 - {x_2}^2$ we have $Q = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ so since Q is negative definite the follower function becomes concave thus we can convert the LFQBLPP into single level programming problem with complementary constraint.

Step 1:- Consider the above LFQBLPP to find the KKT condition for the second level (follower) problem. For any given value of x_1, we define the Lagrangian function for the follower.

$$L(x_1,x_2,\lambda) = f(x_1,x_2) + \lambda^T g(x_1,x_2) \tag{5.1.1}$$

Where,$f(x_1,x_2) = 2x_1 - 3x_2 - {x_1}^2 - {x_2}^2$ and $g(x_1,x_2) = \begin{cases} 4 - x_1 - 4x_2 \ge 0 \\ 2 - x_1 - x_2 \ge 0 \end{cases}$

Here, we have $\lambda = (\lambda_1, \lambda_2) \ge 0$ is the vector of Lagrangian multipliers, therefore equation (5.1.1) becomes

$$L(x_1,x_2,\lambda_1,\lambda_2) = 2x_1 - 3x_2 - {x_1}^2 - {x_2}^2 + \lambda_1(4 - x_1 - 4x_2) + \lambda_2(2 - x_1 - x_2).$$

By applying the K-K-T conditions, we get:

$$\frac{\partial L}{\partial x_2} \le 0 \Rightarrow -3 - 2x_2 - 4\lambda_1 - \lambda_2 \le 0 \tag{5.1.2}$$

$$\begin{cases} \frac{\partial L}{\partial \lambda_1} \ge 0 \ \Rightarrow 4 - x_1 - 4x_2 \ge 0 \\ \frac{\partial L}{\partial \lambda_2} \ge 0 \ \Rightarrow 2 - x_1 - x_2 \ge 0 \end{cases} \tag{5.1.3}$$

$$x_2 \frac{\partial L}{\partial x_2} = 0 \ \Rightarrow x_2(-3 - 2x_2 - 4\lambda_1 - \lambda_2) = 0 \tag{5.1.4}$$

$$\begin{cases} \lambda_1 \frac{\partial L}{\partial \lambda_1} = 0 \ \Rightarrow \lambda_1(4 - x_1 - 4x_2) = 0 \\ \lambda_2 \frac{\partial L}{\partial \lambda_2} = 0 \ \Rightarrow \lambda_2(2 - x_1 - x_2) = 0 \end{cases} \tag{5.1.5}$$

Now introducing slack variables in equation (5.1.2), we get:

$$-2x_2 - 4\lambda_1 - \lambda_2 + u = 3 \tag{5.1.6}$$

Therefore, from equation (5.1.4) and (5.1.6), we get

$$x_2 u = 0. \tag{5.1.7}$$

Also, introducing surplus variables in equation (5.1.3), we get

$$\begin{cases} 4 - x_1 - 4x_2 - y_1 = 0 \\ 2 - x_1 - x_2 - y_2 = 0 \end{cases} \text{Or } \begin{cases} x_1 + 4x_2 + y_1 = 4 \\ x_1 + x_2 + y_2 = 2 \end{cases} \tag{5.1.8}$$

Therefore, from (5.1.5) and (5.1.8), we get

$$\lambda_1 y_1 = 0, \lambda_2 y_2 = 0. \tag{5.1.9}$$

Equations (5.1.2), (5.1.7), (5.1.8) and (5.1.9) gives the KKT conditions corresponding to the lower level objective function. Thus, the given LFQBLPP becomes an LFPPCC. Therefore, an LFPP can be written as:

$$\begin{cases} \max\limits_{x_1} F(x_1, x_2) = \frac{8x_1+3x_2}{5x_1+2x_2+1} \\ subject\ to\ x_1 + 4x_2 + y_1 = 4 \\ \qquad x_1 + x_2 + y_2 = 2 \\ \qquad -2x_2 - 4\lambda_1 - \lambda_2 + u = 3 \\ \qquad x_2 u = 0 \\ \qquad \lambda_1 y_1 = 0 \\ \qquad \lambda_2 y_2 = 0 \\ \qquad x_i, y_i, \lambda_i, u, i = 1,2. \end{cases} \tag{5.1.10}$$

Step 2: Remove the condition $x_2 u = 0$, $\lambda_1 y_1 = 0$ and $\lambda_2 y_2 = 0$ from equation (5.1.10) and solve LFPP. To solve these, first we find an initial basic feasible solution. From equation (5.1.10) all the constraint without the complementary condition before adding the slack variables can be written as follows:

$$\begin{cases} x_1 + 4x_2 \leq 4 \\ x_1 + x_2 \leq 2 \\ -2x_2 - 4\lambda_1 - \lambda_2 \leq 3 \end{cases} \tag{5.1.11}$$

Therefore, equation (5.1.11) is feasible because (0, 0, 0, 0) is a feasible solution. Now find a basic feasible solution $X_B = B^{-1}b$,

where the basic matrix $B = [a_3,\ a_4,\ a_7] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$. Since the basic matrix B is identity the associated basic inverse is also identity matrix, thus $B = B^{-1}$. Therefore $X_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \Rightarrow X_B = \begin{bmatrix} y_1 \\ y_2 \\ u \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ and all the non-basic variable value are zero.

Let $X_1 = (0, 0, 4,2,0,0,3)$. Therefore, the initial basic feasible solution is written as follows:

Table 1: Initial basic feasible solution of problem 5.1.

	$\nabla F(X_1)$	8	3	0	0	0	0	0
V_B	$X_B{}^1$	x_1	x_2	y_1	y_2	λ_1	λ_2	u
y_1	4	1	4	1	0	0	0	0
y_2	2	1	1	0	1	0	0	0
u	3	0	-2	0	0	-4	-1	1
	r	8	3	0	0	0	0	0

Step 3: Compute the vector $r_N{}^T = \nabla_N F(X_s)^T - \nabla_B F(X_s)^T B^{-1} N$, where N is the coefficient matrix of the non-basic variable. So to find the entering variable using $r_N{}^T$, first find $\nabla F(x_1, x_2)$ by calculating the partial derivative of the objective function as:

$$\nabla F(x_1, x_2) = \begin{pmatrix} \frac{x_2+8}{(5x_1+2x_2+1)^2} \\ \frac{-x_1+3}{(5x_1+2x_2+1)^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \nabla F(X_1) = \begin{pmatrix} 8 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$r_N{}^T = (r_1, r_2, r_5, r_6) = \nabla_N F(X_1)^T - \nabla_B F(X_1)^T B^{-1} N$$

$$= (8, 3, 0, 0) - (0, 0, 0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & -4 & -1 \end{bmatrix}$$

$$= (8, 3, 0, 0), \text{ where } N = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -2 & -4 & -1 \end{bmatrix}.$$

Since, all r_N are not less than or equal to zero, then the current point $X_1 = (0, 0, 4,2,0,0,3)$ is not optimal solution of an LFPP. Then go to step 4.

Step 4: Let $r_s = max\{r_i : r_i \geq 0\} = max\{8, 3, 0, 0\} = 8$, therefore x_1 is entering variable. Now determine the basic variable to leave the basis by using minimum ratio test:

$$i.e\; X_{B_r} = \frac{V_{Br}}{Y_{r_j}} = \min_{1 \leq i \leq m+n_2} \left\{ \frac{V_{Bi}}{Y_{ij}} : Y_{ij} > 0 \right\} = min\left\{\frac{4}{1}, \frac{2}{1}\right\} = 2,$$

where $V_{Bi} = B^{-1}b = (4,2,3)^T$ and $Y_{ij} = B^{-1}a_1 = (1,1,0)^T$ and a_1 is the first column of a matrix U whose entries are the coefficients of all basic and non-basic variables in the constraint. Therefore, y_2 is leaving variable. Now we are going to step 5.

Step 5: Replace the leaving basic variable y_2 by the entering variable x_1, then update the table. Thus the second basic feasible solution is given by:

Table 2: The final basic feasible solution of problem 5.1.

	$\nabla F(X_2)$	8/121	1/121	0	0	0	0	0
V_B	${X_B}^2$	x_1	x_2	y_1	y_2	λ_1	λ_2	u
y_1	2	0	3	1	-1	0	0	0
x_1	2	1	1	0	1	0	0	0
u	3	0	-2	0	0	-4	-1	1
	r	0	-7/121	0	-8/121	0	0	0

where $x_1 - Row = \frac{y_2 - Row}{pivotal\ element} = \frac{(2,1,1,0,1,0,0,0,0)}{1} = (2,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)$

$$\begin{aligned} new\ y_1 - Row &= old\ y_1 - Row - (pivotal\ element\ at\ the\ entire\ row)(x_1 - Row) \\ &= (4,1,4,1,0,0,0,0) - 1 \times (2,1,1,0,1,0,0,0) = (2,0,3,1,-1,0,0,0), \\ new\ u - Row &= old\ u - Row - (pivotal\ element\ at\ the\ entire\ row)(x_1 - Row) \\ &= (3,0,-2,0,0,-4,-1,1) - 0 \times (2,1,1,0,1,0,0,0,0) \\ &= (3,0,-2,0,0,-4,-1,1) \text{ and } \nabla F(X_2) = \left(\frac{8}{121}, \frac{1}{121}, 0,0,0,0,0\right). \end{aligned}$$

Let $X_2 = (2,0,2,0,0,0,3)$, now go to step 3:

Step 3: Compute the vector ${r_N}^T = (r_2, r_4, r_5, r_6) = \nabla_N F(X_s)^T - \nabla_B F(X_s)^T B^{-1} N$ where,

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & -4 & -1 \end{pmatrix}, \nabla_B F(X_2)^T = \left(\frac{8}{121}, 0, 0\right) \text{ and}$$

$$\nabla_N F(X_2)^T = \left(\frac{1}{121}, 0, 0, 0\right).$$

$$r_N = \left(\frac{1}{121}, 0,0,0\right) - \left(\frac{8}{121}, 0, 0\right) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & -4 & -1 \end{pmatrix} = \left(\frac{-7}{121}, \frac{-8}{121}, 0, 0\right).$$

Now all $r_N \leq 0$, therefore the above solution X_2 is optimal solution, so go to step 6.

Step 6: Check whether the complementary condition is satisfied or not. $x_2 u = 0$, $\lambda_1 y_1 = 0$, $\lambda_2 y_2 = 0$, as $x_2 = y_2 = \lambda_1 = \lambda_2 = 0$. Therefore, all the complementary conditions are satisfied.

Hence, the solution $X_2 = (2, 0, 2, 0, 0, 0, 3)$ is an optimal solution of the original problem with $F = \frac{16}{11}$ and $f = 0$.

Example 5.2 (Arora and Arora, 2012): Consider the following LFQBLPP.

$$\begin{cases} \underset{x_1}{\text{Max}}\, F(x_1, x_2, x_3) = \frac{2+x_1-2x_2-2x_3}{3+x_1+x_3} \\ \qquad\qquad where\ (x_2, x_3)\ solves \\ \underset{x_2,x_3}{\text{Max}}\, f(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + {x_3}^2 + 2x_1x_2 + 2x_2x_3 - 2x_1 - 7x_2 - 6x_3 \\ \qquad\qquad subject\ to\ x_1 + 2x_2 + x_3 \leq 10 \\ \qquad\qquad x_1 + x_3 \leq 2 \\ \qquad\qquad 3x_1 + x_2 \leq 4 \\ \qquad\qquad x_1, x_2, x_3 \geq 0. \end{cases}$$

Solution:

Step 1:- Consider the above LFQBLPP to find the KKT condition for the second level (follower) problem. For any given value of x_1, we define the Lagrangian function for the follower.

$$L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda^T g(x_1, x_2, x_3) \qquad (5.2.1)$$

Where, $f(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + {x_3}^2 + 2x_1x_2 + 2x_2x_3 - 2x_1 - 7x_2 - 6x_3$ and

$$g(x_1, x_2, x_3) = \begin{cases} 10 - x_1 - 2x_2 - x_3 \geq 0 \\ 2 - x_1 - x_3 \geq 0 \\ 4 - 3x_1 - x_2 \geq 0 \end{cases}$$

Here, we have $\lambda = (\lambda_1, \lambda_2) \geq 0$ is the vector of Lagrangian multipliers, therefore equation (5.2.1) becomes

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = {x_1}^2 + {x_2}^2 + {x_3}^2 + 2x_1x_2 + 2x_2x_3 - 2x_1 - 7x_2 - 6x_3 + \lambda_1(10 - x_1 - 2x_2 - x_3) + \lambda_2(2 - x_1 - x_3) + \lambda_3(4 - 3x_1 - x_2)$$

By applying the KKT conditions, we get:

$$\begin{cases} \frac{\partial L}{\partial x_2} \leq 0 \Rightarrow 2x_2 + 2x_1 + 2x_3 - 7 - 2\lambda_1 - \lambda_3 \leq 0 \\ \frac{\partial L}{\partial x_3} \leq 0 \Rightarrow 2x_3 + 2x_2 - 6 - \lambda_1 - \lambda_2 \leq 0 \end{cases} \qquad (5.2.2)$$

$$\begin{cases} \frac{\partial L}{\partial \lambda_1} \geq 0 \Rightarrow 10 - x_1 - 2x_2 - x_3 \geq 0 \\ \frac{\partial L}{\partial \lambda_2} \geq 0 \Rightarrow 2 - x_1 - x_3 \geq 0 \\ \frac{\partial L}{\partial \lambda_3} \geq 0 \Rightarrow 4 - 3x_1 - x_2 \geq 0 \end{cases} \qquad (5.2.3)$$

$$\begin{cases} x_2 \dfrac{\partial L}{\partial x_2} = 0 \ \Rightarrow x_2(2x_1 + 2x_2 + 2x_3 - 2\lambda_1 - \lambda_3 - 7) = 0 \\ x_3 \dfrac{\partial L}{\partial x_3} = 0 \ \Rightarrow x_3(2x_2 + 2x_3 - \lambda_1 - \lambda_2 - 6) = 0 \end{cases} \tag{5.2.4}$$

$$\begin{cases} \lambda_1 \dfrac{\partial L}{\partial \lambda_1} = 0 \ \Rightarrow \lambda_1(10 - x_1 - 2x_2 - x_3) = 0 \\ \lambda_2 \dfrac{\partial L}{\partial \lambda_2} = 0 \ \Rightarrow \lambda_2(2 - x_1 - x_3) = 0 \\ \lambda_3 \dfrac{\partial L}{\partial \lambda_3} = 0 \ \Rightarrow \lambda_3(4 - 3x_1 - x_2) = 0 \end{cases} \tag{5.2.5}$$

Now introducing slack variables in equation (5.2.2) produce

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 - 2\lambda_1 - \lambda_3 + u_1 = 7 \\ 2x_2 + 2x_3 - \lambda_1 - \lambda_2 + u_2 = 6 \end{cases} \tag{5.2.6}$$

Therefore, from equation (5.2.4) and (5.2.6), we get

$$x_2 u_1 = 0, \ x_3 u_2 = 0. \tag{5.2.7}$$

Also, introducing surplus variables in equation (5.2.3), we get:

$$\begin{cases} 10 - x_1 - 2x_2 - x_3 - y_1 = 0 \\ 2 - x_1 - x_3 - y_2 = 0 \\ 4 - 3x_1 - x_2 - y_3 = 0 \end{cases} \quad \text{Or} \quad \begin{cases} x_1 + 2x_2 + x_3 + y_1 = 10 \\ x_1 + x_3 + y_2 = 2 \\ 3x_1 + x_2 + y_3 = 4 \end{cases} \tag{5.2.8}$$

Therefore, from (5.2.5) and (5.2.8), we get

$$\lambda_1 y_1 = 0, \lambda_2 y_2 = 0, \lambda_3 y_3 = 0. \tag{5.2.9}$$

Equations (5.2.2), (5.2.7), (5.2.8) and (5.2.9) gives the KKT conditions corresponding to the lower level objective function. Thus, the given LFQBLPP becomes an LFPPCC. Therefore, LFPP can be written as:

$$\begin{cases} \underset{x_1}{\text{Max}}\, F(x_1, x_2, x_3) = \frac{2 + x_1 - 2x_2 - 2x_3}{3 + x_1 + x_3} \\ subject\ to\ \ x_1 + 2x_2 + x_3 + y_1 = 10 \\ \qquad x_1 + x_3 + y_2 = 2 \\ \qquad 3x_1 + x_2 + y_3 = 4 \\ \qquad 2x_1 + 2x_2 + 2x_3 - 2\lambda_1 - \lambda_3 + u_1 = 7 \\ \qquad 2x_2 + 2x_3 - \lambda_1 - \lambda_2 + u_2 = 6 \\ \qquad x_2 u_1 = 0, x_3 u_2 = 0, \lambda_1 y_1 = 0, \lambda_2 y_2 = 0, \lambda_3 y_3 = 0 \\ \qquad x_1, x_2, x_3, y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3, u_1, u_2 \geq 0. \end{cases} \tag{5.2.10}$$

Step 2: Remove the condition $x_2 u_1 = 0, x_3 u_2 = 0\ \lambda_1 y_1 = 0, \lambda_2 y_2 = 0$ and $\lambda_3 y_3 = 0$ from equation (5.2.10) and solve LFPP. To solve these, first we find an initial basic feasible solution. From equation (5.2.10) all the constraint without the complementary condition before adding the slack variables can be written as follows:

$$\begin{cases} x_1 + 2x_2 + x_3 & \leq 10 \\ x_1 + x_3 & \leq 2 \\ 3x_1 + x_2 & \leq 4 \\ 2x_1 + 2x_2 + 2x_3 - 2\lambda_1 - \lambda_3 & \leq 7 \\ 2x_2 + 2x_3 - \lambda_1 - \lambda_2 & \leq 6 \end{cases} \tag{5.2.11}$$

Therefore, equation (5.2.11) is feasible because (0, 0, 0, 0, 0, 0) is a feasible solution. Now find a basic feasible solution $\mathrm{X}_B = B^{-1}b$. Where, the basic matrix $B = [a_4, a_5, a_6, a_{10}, a_{11}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 2 \\ 4 \\ 7 \\ 6 \end{bmatrix}$. Since the basic matrix B is identity the associated basic inverse is also identity matrix, thus $B = B^{-1}$. Therefore $\mathrm{X}_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 4 \\ 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 4 \\ 7 \\ 6 \end{bmatrix}$ and all the non-basic variable values are zero. Therefore $\mathrm{X}_1 = (0, 0, 0, 10, 2, 4, 0, 0, 0, 7, 6)$ Thus, the initial basic feasible solution is written as follows:

Table 3: Initial basic feasible solution of problem 5.2.

	$\nabla F(\mathrm{X}_1)$	$\frac{1}{9}$	$\frac{-2}{3}$	$\frac{-8}{9}$	0	0	0	0	0	0	0	0
V_B	$\mathrm{X}_B{}^1$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	10	1	2	1	1	0	0	0	0	0	0	0
y_2	2	1	0	1	0	1	0	0	0	0	0	0
y_3	4	3	1	0	0	0	1	0	0	0	0	0
u_1	7	2	2	2	0	0	0	-2	0	-1	1	0
u_2	6	0	2	2	0	0	0	-1	-1	0	0	1
	r	$\frac{1}{9}$	$\frac{-2}{3}$	$\frac{-8}{9}$	0	0	0	0	0	0	0	0

Step 3: Compute the vector ${r_N}^T = \nabla_N F(\mathrm{X}_s)^T - \nabla_B F(\mathrm{X}_s)^T B^{-1} N$, where N is the coefficient matrix of the non-basic variable. So to find the entering variable using ${r_N}^T$, first find $\nabla F(x_1, x_2, x_3)$ by calculating the partial derivative of the objective function as:

$$\nabla F(x_1, x_2, x_3) = \begin{pmatrix} \frac{2x_2+3x_3+1}{(3+x_1+x_3)^2} \\ \frac{-2x_1-2x_3-6}{(3+x_1+x_3)^2} \\ \frac{-x_1+2x_2-8}{(3+x_1+x_3)^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \nabla F(\mathrm{X}_1) = \begin{pmatrix} \frac{1}{9} \\ \frac{-2}{3} \\ \frac{-8}{9} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$r_N{}^T = (r_1, r_2, r_3, r_7, r_8, r_9) = \nabla_N F(\mathrm{X}_1)^T - \nabla_B F(\mathrm{X}_1)^T B^{-1} N$$

$$= \left(\frac{1}{9}, \frac{-2}{3}, \frac{-8}{9}, 0, 0, 0\right) - (0, 0, 0, 0, 0) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & -2 & 0 & -1 \\ 0 & 2 & 2 & -1 & -1 & 0 \end{bmatrix}$$

$$= \left(\frac{1}{9}, \frac{-2}{3}, \frac{-8}{9}, 0, 0, 0\right), \text{ where } N = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & -2 & 0 & -1 \\ 0 & 2 & 2 & -1 & -1 & 0 \end{bmatrix}.$$

Since, all r_N are not less than or equal to zero, then the current point $\mathrm{X}_1 = (0, 0, 0, 10, 2, 4, 0, 0, 0, 7, 6)$ is not optimal solution of LFPP. Then go to step 4.

Step 4: Let $r_s = max\{r_i : r_i \geq 0\} = max\left\{\frac{1}{9}, 0, 0, 0\right\} = \frac{1}{9}$, therefore x_1 is entering variable. Now determine the basic variable to leave the basis by using minimum ratio test:

$$i.e\ \mathrm{X}_{B_r} = \frac{\mathrm{V}_{B_r}}{\mathrm{Y}_{r_j}} = \min_{1 \leq i \leq m+n_2} \left\{\frac{\mathrm{V}_{Bi}}{\mathrm{Y}_{ij}} : \mathrm{Y}_{ij} > 0\right\} = min\left\{\frac{10}{1}, \frac{2}{1}, \frac{4}{3}, \frac{7}{2}\right\} = \frac{4}{3},$$

where $\mathrm{V}_{Bi} = B^{-1}b = (10, 2, 4, 7, 6)^T$ and $\mathrm{Y}_{ij} = B^{-1}a_1 = (1, 1, 3, 2, 0)^T$ and a_1 is the first column of a matrix U whose entries are the coefficients of all basic and non-basic variables in the constraint. Therefore, y_3 is leaving variable.

Step 5: Replace the leaving basic variable y_3 by the entering variable x_1, then update the table. Thus the second basic feasible solution is given by:

Table 4: The final basic feasible solution of problem 5.2.

	$\nabla F(\mathrm{X}_2)$	$\frac{9}{169}$	$\frac{-78}{169}$	$\frac{-84}{169}$	0	0	0	0	0	0	0	0
V_B	$\mathrm{X}_B{}^2$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	$26/3$	0	5/3	1	1	0	-1/3	0	0	0	0	0
y_2	$2/3$	0	-1/3	1	0	1	-1/3	0	0	0	0	0
x_1	4/3	1	1/3	0	0	0	1/3	0	0	0	0	0
u_1	$13/3$	0	4/3	2	0	0	-2/3	-2	0	-1	1	0
u_2	6	0	2	2	0	0	0	-1	-1	0	0	1
	r	$\frac{-81}{169}$	$\frac{-108}{169}$	0	0	0	$\frac{-3}{169}$	0	0	0	0	0

where $x_1 - Row = \frac{y_3 - row}{pivotal\ element} = \frac{(4,3,1,0,0,0,1,0,0,0,0,0)}{3} = \left(\frac{4}{3}, 1, \frac{1}{3}, 0, 0, 0, \frac{1}{3}, 0, 0, 0, 0, 0\right)$

$$\begin{aligned} new\ y_1 - Row &= old\ y_1 - Row - (pivotal\ element\ at\ the\ entire\ row)(x_1 - Row) \\ &= (10, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0) - 1\left(\frac{4}{3}, 1, \frac{1}{3}, 0, 0, 0, \frac{1}{3}, 0, 0, 0, 0, 0\right) \\ &= \left(\frac{26}{3}, 0, \frac{5}{3}, 1, 1, 0, -\frac{1}{3}, 0, 0, 0, 0, 0\right). \end{aligned}$$

Using the same process we get:

$new\ y_2 - Row = \left(\frac{2}{3}, 0, \frac{-1}{3}, 1, 0, 1, \frac{-1}{3}, 0, 0, 0{,}0{,}0\right),$

$new\ u_1 - Row = \left(\frac{13}{3}, 0, \frac{4}{3}, 2, 0, 0, \frac{-2}{3}, -2, 0, -1, 1, 0\right),$

$new\ u_2 - Row = (6, 0, 2, 2, 0, 0, 0, -1, -1, 0{,}0, 1)$ and

$\nabla F(\mathrm{X}_2) = (\frac{9}{169}, \frac{-78}{169}, \frac{-84}{169}, 0, 0, 0, 0, 0, 0, 0, 0).$

Let $\mathrm{X}_2 = \left(\frac{4}{3}, 0{,}0, \frac{26}{3}, \frac{2}{3}, 0, 0, 0, 0, \frac{13}{3}, 6\right)$, now go to step 3.

Step 3: Compute the vector $r_N{}^T = (r_1, r_2, r_7, r_8, r_9, r_{10}) = \nabla_N F(\mathrm{X}_s)^T - \nabla_B F(\mathrm{X}_s)^T B^{-1} N$ where,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 5/3 & 1 & -1/3 & 0 & 0 & 0 \\ -1/3 & 1 & -1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 4/3 & 2 & -2/3 & -2 & 0 & -1 \\ 2 & 2 & 0 & -1 & -1 & 0 \end{pmatrix}, \nabla_B F(\mathrm{X}_2)^T = (\frac{9}{169}, 0, 0{,}0{,}0\ ),$$

$\nabla_N F(\mathrm{X}_2)^T = (\frac{-78}{169}, \frac{-84}{169}, 0, 0, 0, 0).$

$$r_N = \left(\frac{-78}{169}, \frac{-84}{169}, 0, 0, 0, 0\right) - \left(0, \frac{9}{169}, 0,0,0\right)\begin{pmatrix} 5/3 & 1 & -1/3 & 0 & 0 & 0 \\ -1/3 & 1 & -1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 4/3 & 2 & -2/3 & -2 & 0 & -1 \\ 2 & 2 & 0 & -1 & -1 & 0 \end{pmatrix}$$

$$= \left(-\frac{81}{169}, -\frac{108}{169}, -\frac{3}{169}, 0, 0, 0\right).$$

Now all $r_N \leq 0$, therefore the above solution X_2 is optimal solution, so go to step 6.

Step 6: Check whether the complementary condition is satisfied or not. $x_2 u_1 = 0,\ x_3 u_2 = 0,\ \lambda_1 y_1 = 0,\ \lambda_2 y_2 = 0,\ \lambda_3 y_3 = 0$ as $x_2 = x_3 = y_3 = \lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore, all the complementary conditions are satisfied.

Hence, the solution $\mathrm{X}_2 = \left(\frac{4}{3}, 0,0, \frac{26}{3}, \frac{2}{3}, 0, 0, 0, 0, \frac{13}{3}, 6\right)$ is an optimal solution of the original problem with $F = \frac{10}{13}$ and $f = \frac{-8}{9}$.

Example 5.3 (Arora and Arora, 2011): Consider the following LFQBLPP.

$$\begin{cases} \max\limits_{x_1} F(x_1, x_2, x_3) = \frac{2 - x_1 - x_2 + 2x_3}{4 + x_1 + 3x_3} \\ \qquad\qquad\qquad where\ (x_2, x_3)\ solves \\ \max\limits_{x_2, x_3} f(x_1, x_2, x_3) = {x_1}^2 + 2{x_2}^2 + {x_3}^2 + 2x_1x_2 + 2x_2x_3 + 2x_1 - x_2 - 8x_3 \\ \qquad subject\ to \quad x_1 + 2x_2 + x_3 \leq 10 \\ \qquad\qquad\qquad x_1 - x_3 \leq 2 \\ \qquad\qquad\qquad 3x_1 + 5x_2 \leq 4 \\ \qquad\qquad\qquad x_1, x_2, x_3 \geq 0. \end{cases}$$

By using the usual method of converting maximization into minimization we get the following problem.

$$\begin{cases} \min\limits_{x_1} F(x_1, x_2, x_3) = \frac{x_1 + x_2 - 2x_3 - 2}{4 + x_1 + 3x_3} \\ \qquad\qquad\qquad where\ (x_2, x_3)\ solves \\ \min\limits_{x_2, x_3} f(x_1, x_2, x_3) = -{x_1}^2 - 2{x_2}^2 - {x_3}^2 - 2x_1x_2 - 2x_2x_3 - 2x_1 + x_2 + 8x_3 \\ \qquad subject\ to \quad x_1 + 2x_2 + x_3 \leq 10 \\ \qquad\qquad\qquad x_1 - x_3 \leq 2 \\ \qquad\qquad\qquad 3x_1 + 5x_2 \leq 4 \\ \qquad\qquad\qquad x_1, x_2, x_3 \geq 0. \end{cases}$$

Solution:

Step 1:- Consider the above LFQBLPP to find the KKT condition for the second level (follower) problem. For any given value of x_1, we define the Lagrangian function as follows:

$$L(x_1, x_2, x_3, \lambda) = f(x_1, x_2, x_3) + \lambda^T g(x_1, x_2, x_3). \tag{5.3.1}$$

Where, $f(x_1, x_2, x_3) = -{x_1}^2 - 2{x_2}^2 - {x_3}^2 - 2x_1x_2 - 2x_2x_3 - 2x_1 + x_2 + 8x_3$ and

$$g(x_1, x_2, x_3) = \begin{cases} x_1 + 2x_2 + x_3 - 10 \leq 0 \\ x_1 - x_3 - 2 \leq 0 \\ 3x_1 + 5x_2 - 4 \leq 0 \end{cases}$$

Here, we have $\lambda = (\lambda_1, \lambda_2, \lambda_3) \geq 0$ is the vector of Lagrangian multipliers, therefore the Lagrangian function of equation (5.3.1) becomes

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = -{x_1}^2 - 2{x_2}^2 - {x_3}^2 - 2x_1x_2 - 2x_2x_3 - 2x_1 + x_2 + 8x_3 + \lambda_1(x_1 + 2x_2 + x_3 - 10) + \lambda_2(x_1 - x_3 - 2) + \lambda_3(3x_1 + 5x_2 - 4).$$

By applying the KKT conditions, we get:

$$\begin{cases} \frac{\partial L}{\partial x_2} \geq 0 \Rightarrow -2x_1 - 4x_2 - 2x_3 + 2\lambda_1 + 5\lambda_3 + 1 \geq 0 \\ \frac{\partial L}{\partial x_3} \geq 0 \ \Rightarrow -2x_2 - 2x_3 + \lambda_1 - \lambda_2 + 8 \geq 0 \end{cases} \tag{5.3.2}$$

$$\begin{cases} \frac{\partial L}{\partial \lambda_1} \leq 0 \ \Rightarrow x_1 + 2x_2 + x_3 - 10 \leq 0 \\ \frac{\partial L}{\partial \lambda_2} \leq 0 \ \Rightarrow x_1 - x_3 - 2 \leq 0 \\ \frac{\partial L}{\partial \lambda_3} \leq 0 \ \Rightarrow 3x_1 + 5x_2 - 4 \leq 0 \end{cases} \tag{5.3.3}$$

$$\begin{cases} x_2 \frac{\partial L}{\partial x_2} = 0 \ \Rightarrow x_2(-2x_1 - 4x_2 - 2x_3 + 2\lambda_1 + 5\lambda_3 + 1) = 0 \\ x_3 \frac{\partial L}{\partial x_3} = 0 \ \Rightarrow x_3(-2x_2 - 2x_3 + \lambda_1 - \lambda_2 + 8) = 0 \end{cases} \tag{5.3.4}$$

$$\begin{cases} \lambda_1 \frac{\partial L}{\partial \lambda_1} = 0 \ \Rightarrow \lambda_1(x_1 + 2x_2 + x_3 - 10) = 0 \\ \lambda_2 \frac{\partial L}{\partial \lambda_2} = 0 \ \Rightarrow \lambda_2(x_1 - x_3 - 2) = 0 \\ \lambda_3 \frac{\partial L}{\partial \lambda_3} = 0 \ \Rightarrow \lambda_3(3x_1 + 5x_2 - 4) = 0 \end{cases} \tag{5.3.5}$$

Now introducing surplus variables in equation (5.3.2) we get:

$$\begin{cases} -2x_1 - 4x_2 - 2x_3 + 2\lambda_1 + 5\lambda_3 - u_1 = -1 \\ -2x_2 - 2x_3 + \lambda_1 - \lambda_2 - u_2 = -8 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_1 + 4x_2 + 2x_3 - 2\lambda_1 - 5\lambda_3 + u_1 = 1 \\ 2x_2 + 2x_3 - \lambda_1 + \lambda_2 + u_2 = 8 \end{cases} \tag{5.3.6}$$

Therefore, from equation (5.3.4) and (5.3.6) produce

$$x_2u_1 = 0 \ and \ x_3u_2 = 0. \tag{5.3.7}$$

Also, introducing slack variables in equation (5.3.3), we get

$$\begin{cases} x_1 + 2x_2 + x_3 - 10 + y_1 = 0 \\ x_1 - x_3 - 2 + y_2 = 0 \\ 3x_1 + 5x_2 - 4 + y_3 = 0 \end{cases} \quad \text{Or} \quad \begin{cases} x_1 + 2x_2 + x_3 + y_1 = 10 \\ x_1 - x_3 + y_2 = 2 \\ 3x_1 + 5x_2 + y_3 = 4 \end{cases} \tag{5.3.8}$$

Therefore, from (5.3.5) and (5.3.8) we get:

$$\lambda_1 y_1 = 0, \lambda_2 y_2 = 0, \lambda_3 y_3 = 0. \tag{5.3.9}$$

Equations (5.3.2), (5.3.7), (5.3.8) and (5.3.9) gives the KKT conditions corresponding to the lower level objective function. Thus, the given LFQBLPP becomes an LFPPCC. Therefore, LFPP can be written as:

$$\begin{cases} \min\limits_{x_1} F(x_1, x_2, x_3) = \frac{x_1+x_2-2x_3-2}{4+x_1+3x_3} \\ subject\ to\ \ x_1 + 2x_2 + x_3 + y_1 = 10 \\ \qquad x_1 - x_3 + y_2 = 2 \\ \qquad 3x_1 + 5x_2 + y_3 = 4 \\ \qquad 2x_1 + 4x_2 + 2x_3 - 2\lambda_1 - 5\lambda_3 + u_1 = 1 \\ \qquad 2x_2 + 2x_3 - \lambda_1 + \lambda_2 + u_2 = 8 \\ \quad x_2 u_1 = 0, x_3 u_2 = 0, \lambda_1 y_1 = 0, \lambda_2 y_2 = 0, \lambda_3 y_3 = 0 \\ \qquad x_i, y_i, \lambda_i, u_j, i = 1,2,3\ and\ j = 1,2. \end{cases} \tag{5.3.10}$$

Step 2: Remove the condition $x_2 u_1 = 0,\ x_3 u_2 = 0,\ \lambda_1 y_1 = 0,\ \lambda_2 y_2 = 0$ and $\lambda_3 y_3 = 0$ from equation (5.3.10) and solve LFPP. To solve these, first we find an initial basic feasible solution and express the basic variables in terms of non-basic variables. From equation (5.3.10) all the constraint without the complementary condition before adding the slack variables can be written as follows:

$$\begin{cases} x_1 + 2x_2 + x_3 \le 10 \\ x_1 - x_3 \le 2 \\ 3x_1 + 5x_2 \le 4 \\ 2x_1 + 4x_2 + 2x_3 - 2\lambda_1 - 5\lambda_3 \le 1 \\ 2x_2 + 2x_3 - \lambda_1 + \lambda_2 \le 8 \end{cases} \tag{5.3.11}$$

Therefore, equation (11) is feasible because (0, 0, 0, 0, 0, 0) is a feasible solution. Now find a basic feasible solution $\mathrm{X}_B = B^{-1}b$.

Where, the basic matrix $B = [a_4,\ a_5,\ a_6,\ a_{10},\ a_{11}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 2 \\ 4 \\ 1 \\ 8 \end{bmatrix}$. Since the basic matrix B is identity the associated basic inverse is also identity matrix, thus $B = B^{-1}$. Therefore $\mathrm{X}_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 4 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 4 \\ 1 \\ 8 \end{bmatrix} \Rightarrow \mathrm{X}_B = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 4 \\ 1 \\ 8 \end{bmatrix}$ and all the non-basic variable value are zero. Therefore, $X_1 = (0,0,0,10,2,4,0,0,0,1,8)$. Hence, the initial basic feasible solution is written as follows:

Table 5: Initial basic feasible solution of problem 5.3.

	$\nabla F(X_1)$	$\frac{3}{8}$	$\frac{1}{4}$	$-\frac{1}{8}$	0	0	0	0	0	0	0	0
V_B	${X_B}^1$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	10	1	2	1	1	0	0	0	0	0	0	0
y_2	2	1	0	-1	0	1	0	0	0	0	0	0
y_3	4	3	5	0	0	0	1	0	0	0	0	0
u_1	1	2	4	2	0	0	0	-2	0	-5	1	0
u_2	8	0	2	2	0	0	0	-1	1	0	0	1
	r	$\frac{3}{8}$	$\frac{1}{4}$	$-\frac{1}{8}$	0	0	0	0	0	0	0	0

Step 3: Compute the vector ${r_N}^T = \nabla_N F(X_s)^T - \nabla_B F(X_s)^T B^{-1} N$, where N is the coefficient matrix of the non-basic variable.

$$\nabla F(x_1,x_2,x_3) = \begin{pmatrix} \frac{-x_2+5x_3+6}{(4+x_1+3x_3)^2} \\ \frac{1}{4+x_1+3x_3} \\ \frac{-2-5x_1-3x_2}{(4+x_1+3x_3)^2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \nabla F(X_1) = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{4} \\ -\frac{1}{8} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & -2 & 0 & -5 \\ 0 & 2 & 2 & -1 & 1 & 0 \end{pmatrix}$$

$${r_N}^T = (r_1, r_2, r_3, r_7, r_8, r_9) = \nabla_N F(X_1)^T - \nabla_B F(X_1)^T B^{-1} N$$

$$= \left(\frac{3}{8}, \frac{1}{4}, -\frac{1}{8}, 0, 0, 0\right) - (0,0,0,0,0) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & -2 & 0 & -5 \\ 0 & 2 & 2 & -1 & 1 & 0 \end{pmatrix}$$

$$= (3/8, 1/4, -1/8, 0, 0, 0).$$

Since, all r_N are not greater than or equal to zero, then the current point $X_1 = (0,0,0,10,2,4,0,0,0,1,8)$ is not optimal solution of LFPP. Then go to step 4.

Step 4: Let $r_s = min\{r_i : r_i \le 0\} = min\left\{-\frac{1}{8}, 0, 0, 0\right\} = -\frac{1}{8}$, therefore x_3 is entering variable. Now determine the basic variable to leave the basis by using minimum ratio test:

$$i.e\ X_{B_r} = \frac{V_{Br}}{Y_{r_j}} = \min_{1 \le i \le m+n_2} \left\{\frac{V_{Bi}}{Y_{ij}} : Y_{ij} > 0\right\} = min\left\{\frac{10}{1}, \frac{1}{2}, \frac{8}{2}\right\} = \frac{1}{2},$$

where $V_{Bi} = B^{-1}b = (10, 2, 4, 1, 8)^T$ and $Y_{ij} = B^{-1}a_3 = (1, -1, 0, 2, 2)^T$ and a_3 is the third column of a matrix U whose entries are the coefficients of all basic and non-basic variables in the constraint. Therefore, u_1 is leaving variable.

Step 5: Replace the leaving basic variable u_1 by the entering variable x_3, then update the table. Thus the second basic feasible solution is given by:

Table 6: The second basic feasible solution of problem 5.3.

	$\nabla F(X_2)$	$\frac{34}{121}$	$\frac{2}{11}$	$-\frac{8}{121}$	0	0	0	0	0	0	0	0
V_B	${X_B}^2$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	19/2	0	0	0	1	0	0	1	0	5/2	-1/2	0
y_2	5/2	2	2	0	0	1	0	-1	0	-5/2	1/2	0
y_3	4	3	5	0	0	0	1	0	0	0	0	0
x_3	1/2	1	2	1	0	0	0	-1	0	-5/2	1/2	0
u_2	7	-2	-2	0	0	0	0	1	1	5	-1	1
	r	$\frac{42}{121}$	$\frac{38}{121}$	0	0	0	0	$\frac{-8}{121}$	0	$\frac{-20}{121}$	$\frac{4}{121}$	0

where x_3-$Row = \frac{u_1-row}{pivotal\ element} = \frac{(1,2,4,2,0,0,0,-2,0,-5,1,0)}{2} = \left(\frac{1}{2}, 1,2, 1, 0, 0, 0, -1, 0, -\frac{5}{2}, \frac{1}{2}, 0\right)$.

$$new\ y_1 - Row = old\ y_1 - Row - (pivotal\ element\ at\ the\ entire\ row)(x_3 - Row),$$

$$= \left(\frac{19}{2}, 0, 0, 0, 1, 0, 0, 1, 0, \frac{5}{2}, -\frac{1}{2}, 0\right).$$

Using the same procedure we get the following result

$new\ y_2 - Row\ = \left(\frac{5}{2}, 2, 2, 0, 0, 1, 0, -1, 0, -\frac{5}{2}, \frac{1}{2}, 0\right)$,

$new\ y_3 - Row\ = (4, 3, 5, 0, 0,0,1, 0, 0, 0, 0, 0)$ and

$new\ u_2 - Row = (7, -2, -2, 0, 0,0,0, 1, 1, 5, -1, 1)$.

$\nabla F(\mathrm{X}_2) = \left(\frac{34}{121}, \frac{2}{11}, -\frac{8}{121}, 0, 0, 0, 0, 0, 0, 0, 0\right)$.

Let $\mathrm{X}_2 = \left(0, 0, \frac{1}{2}, \frac{19}{2}, \frac{5}{2}, 4, 0, 0, 0, 0, 7\right)$, now go to step 3.

Step 3: Compute the vector ${r_N}^T = (\ r_1, r_2, r_7, r_8, r_9, r_{10}) = \nabla_N F(\mathrm{X}_s)^T - \nabla_B F(\mathrm{X}_s)^T B^{-1} N$,

where $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 0 & 1 & 0 & 5/2 & -1/2 \\ 2 & 2 & -1 & 0 & -5/2 & 1/2 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & -5/2 & 1/2 \\ -2 & -2 & 1 & 1 & 5 & -1 \end{pmatrix}$,

$\nabla_B F(\mathrm{X}_2)^T = \left(-\frac{8}{121}, 0, 0,0,0\right)$ and $\nabla_N F(\mathrm{X}_2)^T = \left(\frac{34}{121}, \frac{2}{11}, 0, 0, 0, 0\right)$.

$$r_N = \left(\frac{34}{121}, \frac{2}{11}, 0, 0, 0, 0\right) - (-\frac{8}{121}, 0, 0,0,0) \begin{pmatrix} 1 & 2 & -1 & 0 & -5/2 & 1/2 \\ 0 & 0 & 1 & 0 & 5/2 & -1/2 \\ 2 & 2 & -1 & 0 & -5/2 & 1/2 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 1 & 5 & -1 \end{pmatrix}$$

$$= \left(\frac{42}{121}, \frac{38}{121}, -\frac{8}{121}, 0, -\frac{20}{121}, \frac{4}{121}\right).$$

Since, all r_N are not greater than or equal to zero, then the current point $\mathrm{X}_2 = (0,0,\frac{1}{2},\frac{19}{2},\frac{5}{2},4,0,0,0,0,7)$ is not optimal solution of LFPP. Then go to step 4.

Step 4: Let $r_s = \min\left\{-\frac{8}{121}, 0, -\frac{20}{121}\right\} = -\frac{20}{121}$ therefore,λ_3 enters. Thus by using the minimum ratio test we have that $\min\left\{\frac{19}{5}, \frac{7}{5}\right\} = \frac{7}{5}$. *i.e.* u_2 departs.

Step 5: Replace the leaving basic variable u_2 by the entering variable λ_3, then update the table. Thus the third basic feasible solution is given as follows:

Table 7: The third basic feasible solution of problem 5.3.

	$\nabla F(\mathrm{X}_3)$	$\frac{13}{128}$	$\frac{1}{16}$	$-\frac{1}{128}$	0	0	0	0	0	0	0	0
V_B	${\mathrm{X}_B}^3$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	6	1	1	0	1	0	0	1/2	-1/2	0	0	-1/2
y_2	6	1	1	0	0	1	0	-1/2	½	0	0	1/2
y_3	4	3	5	0	0	0	1	0	0	0	0	0
x_3	4	0	1	1	0	0	0	-1/2	1/2	0	0	1/2
λ_3	7/5	-2/5	-2/5	0	0	0	0	1/5	1/5	1	-1/5	1/5
	r	$\frac{13}{128}$	$\frac{9}{128}$	0	0	0	0	$\frac{-1}{256}$	$\frac{1}{256}$	0	0	$\frac{1}{256}$

Using the same procedure as we did before we get the following values:

$\lambda_3 - Row = \left(\frac{7}{5}, -\frac{2}{5}, -\frac{2}{5}, 0,0,0,0, \frac{1}{5}, \frac{1}{5}, 1, -\frac{1}{5}, \frac{1}{5}\right),$

$new\ y_3 - Row = (4,3,5,0,0,0,1,0,0,0,0,0),$

$new\ x_3 - Row = \left(4,0,1,1,0,0,0,-\frac{1}{2}, \frac{1}{2}, 0,0, \frac{1}{2}\right),$

$new\ y_2 - Row = \left(6,1,1,0,0,1,0, \frac{-1}{2}, \frac{1}{2}, 0,0, \frac{1}{2}\right)$ and

$y_1 - Row = \left(6, 1, 1, 0, 1, 0, 0, \frac{1}{2}, -\frac{1}{2}, 0, 0, -\frac{1}{2}\right).$

Let $\mathrm{X}_3 = \left(0, 0, 4, 6, 6, 4, 0, 0, \frac{7}{5}, 0, 0\right)$, now go to step 3.

Step 3: Compute the vector ${r_N}^T = (r_1, r_2, r_7, r_8, r_{10}, r_{11}) = \nabla_N F(\mathrm{X}_s)^T - \nabla_B F(\mathrm{X}_s)^T B^{-1} N,$

where $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 1 & 1/2 & -1/2 & 0 & -1/2 \\ 1 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ -2/5 & -2/5 & 1/5 & 1/5 & -1/5 & 1/5 \end{pmatrix}$,

$\nabla_B F(\mathrm{X}_3)^T = \left(-\frac{1}{128}, 0, 0,0,0\right)$, $\nabla_N F(\mathrm{X}_3)^T = \left(\frac{13}{128}, \frac{1}{16}, 0, 0, 0, 0\right).$

$$\Rightarrow r_N = \left(\frac{13}{128}, \frac{1}{16}, 0, 0, 0, 0\right) - \left(-\frac{1}{128}, 0, 0,0,0\right)\begin{pmatrix} 0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 1 & 1 & 1/2 & -1/2 & 0 & -1/2 \\ 1 & 1 & -1/2 & 1/2 & 0 & 1/2 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ -2/5 & -2/5 & 1/5 & 1/5 & -1/5 & 1/5 \end{pmatrix}$$

$$= \left(\frac{13}{128}, \frac{9}{128}, -\frac{1}{256}, \frac{1}{256}, 0, \frac{1}{256}\right).$$

Since, all r_N are not greater than or equal to zero, then the current point $\mathrm{X}_3 = (0, 0, 4, 6, 6, 4, 0, 0, 7/5, 0, 0)$ is not optimal solution of LFPP. Then go to step 4.

Step 4: Let $r_s = \min\left\{-\frac{1}{256}, 0\right\} = -\frac{1}{256}$, therefore λ_1 enters and using minimum ratio test we get: $\min\left\{\frac{6}{1/2}, \frac{7/5}{1/5}\right\} = \min\{12, 7\} = 7 \Rightarrow \lambda_3$ departs.

Step 5: Replace the leaving basic variable λ_3 by the entering variable λ_1, then update the table. Thus the fourth basic feasible solution is given as follows:

Table 8: The fourth basic feasible solution of problem 5.3.

	$\nabla F(\mathrm{X}_4)$	$\frac{174}{2809}$	$\frac{2}{53}$	$-\frac{8}{2809}$	0	0	0	0	0	0	0	0
V_B	$\mathrm{X}_B{}^4$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
y_1	$5/2$	2	2	0	1	0	0	0	-1	-5/2	1/2	-1
y_2	$19/2$	0	0	0	0	1	0	0	1	5/2	-1/2	1
y_3	4	3	5	0	0	0	1	0	0	0	0	0
x_3	15/2	-1	0	1	0	0	0	0	1	5/2	-1/2	1
λ_1	7	-2	-2	0	0	0	0	1	1	5	-1	1
	r	$\frac{166}{2809}$	$\frac{2}{53}$	0	0	0	0	0	$\frac{8}{2809}$	$\frac{20}{2809}$	$\frac{-4}{2809}$	$\frac{8}{2809}$

Following the above procedure we get:

$\lambda_1 - Row = (7, -2, -2,0,0,0,0,1,1,5, -1,1),$

$newy_2 - Row \ = \left(\frac{19}{2}, 0,0,0,0,1,0,0,1, \frac{5}{2}, -\frac{1}{2}, 1\right),$

$newy_3 - Row = (4,3,5,0,0,0,1,0,0,0,0,0),$

$new\ x_3 - Row = \left(\frac{15}{2}, -1,0,1,0,0,0,0,1, \frac{5}{2}, -\frac{1}{2}, 1\right)$ and

$new\ y_1 - Row = \left(\frac{5}{2}, 2, 2, 0, 1,0,0,0, -1, \frac{-5}{2}, \frac{1}{2}, -1\right)$.

Let $X_4 = \left(0, 0, \frac{15}{2}, \frac{5}{2}, \frac{19}{2}, 4, 7, 0,0,0, 0\right)$, now go to step 3.

Step 3: Compute the vector ${r_N}^T = (r_1, r_2, r_8, r_9, r_{10}, r_{11}) = \nabla_N F(X_s)^T - \nabla_B F(X_s)^T B^{-1} N$, where $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 2 & 2 & -1 & -5/2 & 1/2 & -1 \\ 0 & 0 & 1 & 5/2 & -1/2 & 1 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 5/2 & -1/2 & 1 \\ -2 & -2 & 1 & 5 & -1 & 1 \end{pmatrix}$,

$\nabla_N F(X_4)^T = \left(\frac{174}{2809}, \frac{2}{53}, 0, 0, 0, 0\right)$, $\nabla_B F(X_4)^T = \left(-\frac{8}{2809}, 0, 0,0,0\right)$.

$$\Rightarrow r_N = \left(\frac{174}{2809}, \frac{2}{53}, 0, 0, 0, 0\right) - \left(-\frac{8}{2809}, 0, 0,0,0\right) \begin{pmatrix} -1 & 0 & 1 & 5/2 & -1/2 & 1 \\ 2 & 2 & -1 & -5/2 & 1/2 & -1 \\ 0 & 0 & 1 & 5/2 & -1/2 & 1 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ -2 & -2 & 1 & 5 & -1 & 1 \end{pmatrix}$$

$$= \left(\frac{166}{2809}, \frac{2}{53}, \frac{8}{2809}, \frac{20}{2809}, -\frac{4}{2809}, \frac{8}{2809}\right).$$

Since, all r_N are not greater than or equal to zero, then the current point $X_4 = \left(0, 0, \frac{15}{2}, \frac{5}{2}, \frac{19}{2}, 4, 7, 0,0,0, 0\right)$ is not optimal solution of LFPP. Then go to step 4.

Step 4: Let $r_s = \min\left\{-\frac{4}{2809}\right\} = -\frac{4}{2809}$, therefore u_1 enters and using minimum ratio test we get: $min\left\{\frac{5/2}{1/2}\right\} = \min\{\ 5\ \} = 5 \Rightarrow y_1$ departs.

Step 5: Replace the leaving basic variable y_1 by the entering variable u_1, then update the table. Thus the fifth basic feasible solution is given as follows:

Table 9: The final basic feasible solution of problem 5.3.

	$\nabla F(X_5)$	$\frac{14}{289}$	$\frac{1}{34}$	$-\frac{1}{578}$	0	0	0	0	0	0	0	0
V_B	${X_B}^5$	x_1	x_2	x_3	y_1	y_2	y_3	λ_1	λ_2	λ_3	u_1	u_2
u_1	5	4	4	0	2	0	0	0	-2	-5	1	-2
y_2	12	2	2	0	1	1	0	0	0	0	0	0
y_3	4	3	5	0	0	0	1	0	0	0	0	0
x_3	10	1	2	1	1	0	0	0	0	0	0	0
λ_1	12	2	2	0	2	0	0	1	-1	0	0	-1
	r	$\frac{29}{578}$	$\frac{19}{578}$	0	$\frac{1}{578}$	0	0	0	0	0	0	0

Following the above procedure we get:

$u_1 - Row = (5, 4, 4, 0, 2, 0, 0, 0, -2, -5, 1, -2)$,

$new\ y_2 - Row\ = (12, 2,2, 0,1,1, 0,0,0,0,0,0)$,

$new\ y_3 - Row = (4, 3, 5, 0, 0, 0, 1, 0, 0, 0, 0, 0)$,

$new\ x_3 - Row = (10,1, 2, 1, 1,0,0,0,0,0,0,0)$ and

$new\ \lambda_1 - Row = (12, 2, 2, 0, 2, 0, 0,1, -1, 0, 0, -1)$.

Let $X_5 = (0, 0, 10, 0, 12, 4,12,0,0,5,0)$ now go to step 3.

Step 3: Compute the vector ${r_N}^T = (r_1, r_2, r_4, r_8, r_9, r_{11}) = \nabla_N F(X_s)^T - \nabla_B F(X_s)^T B^{-1} N$ where

$$, B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, N = \begin{pmatrix} 4 & 4 & 2 & -2 & -5 & -2 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & -1 & 0 & -1 \end{pmatrix},$$

$\nabla_N F(X_5)^T = \left(\frac{14}{289}, \frac{1}{34}, 0, 0, 0, 0\right)$, $\nabla_B F(X_5)^T = \left(-\frac{1}{578}, 0, 0,0,0\right)$.

$$\Rightarrow r_N = \left(\frac{14}{289}, \frac{1}{34}, 0, 0, 0, 0\right) - \left(-\frac{1}{578}, 0, 0,0,0\right) \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & -1 & 0 & -1 \\ 4 & 4 & 2 & -2 & -5 & -2 \end{pmatrix}$$

$$= \left(\frac{29}{578}, \frac{19}{578}, \frac{1}{578}, 0,0,0\right).$$

Now all $r_N \geq 0$, therefore the above solution X_5 is optimal solution, so go to step 6.

Step 6: Check whether the complementary condition is satisfied or not. $x_2u_1 = 0$, $x_3u_2 = 0$, $\lambda_1 y_1 = 0$, $\lambda_2 y_2 = 0$, $\lambda_3 y_3 = 0$ as $x_2 = u_2 = y_1 = \lambda_2 = \lambda_3 = 0$. Therefore, all the complementary conditions are satisfied.

Hence, the solution $X_5 = (0, 0, 10, 0, 12, 4,12, 0,0,5, 0)$ is an optimal solution of the original problem with $F = -\frac{11}{17}$ and $f = -20$.

Table 10: Comparison of the best solutions for Example 5.1 and Example 5.2.

The compare of the results by the proposed algorithm in this paper and the results in the references is as follows:

	Best solution by an MSM		Best solution by the references	
Example 5.1	$(x, y) = (2,0)$	$F = \frac{16}{11}$ ≈ 1.455	$(x, y) = \left(\frac{4}{3}, \frac{2}{3}\right)$	$F = \frac{38}{27}$ ≈ 1.407
Example 5.2	$(x_1, x_2, x_3) = \left(\frac{4}{3}, 0, 0\right)$	$F = \frac{10}{13}$ ≈ 0.769	$(x_1, x_2, x_3) = (1, 0, 0)$	$F = \frac{3}{4}$ ≈ 0.75

As we did in the above we solve all the two problems using the proposed modified simplex algorithm and we compare the optimal solution in Table 10. According to table 10 in the above, it declares that the best solutions by the proposed algorithm has a better optimal solution in less iteration. It can be seen that the proposed algorithm is efficient and feasible from the result.

6. SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1. Summary

Bi-level programming problem is an optimization problem which involves two optimization problems where the constraint region of the upper level (leader) problem is implicitly determined by the lower level (follower) optimization problem. This project considered the LFQBLPP wherein the leader function is linear fractional and the follower one is convex quadratic with the common linear constraints. The Modified Simplex algorithm is presented in order to solve the LFQBLPP. Firstly, the second level (follower) problem must be convex. Since the lower level problem is a convex optimization problem so that the optimal solution to the second level problem is unique thus, the KKT necessary and sufficient optimality conditions are satisfied. If these conditions are used to replace the lower-level problem, a so-called an equivalent single level LFPPCC arises by converting the constrained second level (follower) problem into the unconstrained problem by using Lagrangian function method. This problem is most often used to replace the optimization problem. Unfortunately, both problems are equivalent only if global optima are searched for. A local optimal solution of the mathematical program with equilibrium constraint is in general not related to a locally optimal solution of the bilevel problem. When the LFPP was solved with complementary condition using the proposed modified simplex algorithm, the complementary conditions must be removed and after the optimal solution is reached it was checked that whether the complementary condition is satisfied or not. Finally, it is shown that the proposed method is efficient by solving some examples.

6.2. Conclusion

In this project, the LFQBLPP has been dealt. The Modified Simplex algorithm was presented in order to solve the LFQBLPP. Since the LFQBLPP is the most challenging of the mathematical programming problem with two-level structure in which the first level (leader) is linear fractional and the second level (follower) one is convex quadratic we have converted into single level programming problem. Since the lower level problem is a convex programming problem, another property of bilevel programs has been described which is taken for granted when dealing with standard optimization problems by transforming the LFQBLPP into an equivalent single-level LFPPCC in order to minimize the complexity. We have proved the optimality condition for the LFPPCC. The transformed problem was solved analytically using the proposed algorithm. It has been observed that solving the present optimization problem of large size with hand is drastically hard. This difficulty can also be stumbled on even for small size problems as the algorithm recycle itself. As a result to overcome such troubles using mathematical software seemed to be inevitable. From the numerical result, the results by the proposed method in this project accord with the results in the reference. So it has been observed that the proposed method has better numerical result in less iteration. Finally, it is concluded that the proposed MSM was efficient for solving the LFPPCC from the computational point of view and quality of solutions.

6.3. Recommendation

In dealing with Linear Fractional/Quadratic Bi-level Programming Problem using the modified simplex method, the following basic recommendations are suggested:

- The difficulty of solving large size problems of the present optimization problem type by hand has been observed. So that developing a computer code using efficient mathematical softwares will make the task easier and it is left as future work.
- The present optimization method can be applied on the other kinds of BLPP such as Quadratic Bi-level programming problem which has Quadratic Objective function on both levels and Quadratic Fractional Bilevel programming problem in which the first level (leader) is Quadratic and the second level (follower) is linear fractional etc.
- The present work can be extend by developing other algorithms like genetic algorithm can solve the Linear Fractional-Quadratic Bi-level programming problem by solving only an equivalent Linear Fractional programming problem with the complementary constraint.

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