NUMERICAL SOLUTION OF NON-LINEAR VOLTERRA-FREDHOLM-HAMMERSTEIN INTEGRAL EQUATIONS BY USING HAAR WAVELET COLLOCATION METHOD

MSc Project

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Numerical Solution of Non -Linear Volterra-Fredholm-Hammerstein Integral Equations by Using Haar Wavelet Collocation Method

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We here by certify that we have read and evaluated this Project titled "**Numerical Solution of Non-Linear Volterra-Fredholm-Hammerstein Integral Equations by Using Haar Wavelet Collocation Method**' prepared under my guidance by Teketel Woldemeskel. All feedback given to the student has been incorporated in the project. Therefore, I recommend that it be submitted as fulfilling the project requirement.

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DEDICATION

I dedicate this project manuscript to my beloved mother Sega Lambebo and brother Delelegn Woldemeskel for their sacrifice throughout my education and support for the success of my life.

STATEMENT OF THE AUTHOR

By my signature below, I declare that this project is my own work. I have followed all ethical and technical principles of scholarship in the preparation, and compilation of this project. Any scholarly matter that is included in the project has been given recognition through citation.

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BIOGRAPHICAL SKETCH

The author was born in September, 1992 in SNNP Regional State, Hadiya Zone, Duna wereda, Chafimera kebele from his mother Sega Lambebo and father Woldemeskel Nuramo. He attended his primary education at Densa Hafule primary schools. Then after, he joined Anisho secondary school and Gimbichu preparatory school to attend his secondary education. He then joined Wachemo University in 2012 and received Bachelor of Science degree in Mathematics in July, 2014. Then he has been employed in SNNP Regional State, Hadiya zone, Duna wereda, Densa Hafule secondary and preparatory school in 2015; and worked for two years in the school, he joined postgraduate program at Haramaya University, College of Natural and Computational Sciences, Department of Mathematics in 2017 to pursue a program of study for M.Sc. degree in Mathematics with specialization in Numerical Analysis.

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ABBREVIATIONS

BIE	Bounary Integral Equations
BVP	Boundary Value Problems
FIE	Fredholm Integral Equation
HWCM	Haar Wavelet Collocation Method
HIE	Hammerstein Integral Equation
IVP	Initial Value Problems
ODE	Ordinary Differential Equations
OM	Operational Matrix
PDE	Partial Differential Equations
VIE	Volterra Integral Equation
VFIE	Volterra-Fredholm Integral Equation
VFHIE	Volterra-Fredholm-Hammerstein Integral Equation

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ABSTRACT

In this project, we discussed numerical solutions of non-linear Volterra-Fredholm-Hammerstein integral equations by using haar wavelet collocation method. Non-linear Volterra-Fredholm-Hammerstein integral equations which can not be easily evaluated analytically. This project was concerned with numerical method (Haar wave collocation method). Haar wavelet collocation method was used to transform non-linear Volterra–Fredholm–Hammerstein integral equations to a system of algebraic equations. Haar wavelet and its operational matrices were utilized to convert the integral equation into a system of algebraic equations and the resulting algebraic equationwere solved by using MATLAB to compute the Haar coefficients.The computational cost for these method was analyzed for examples and the error analysis was done by haar wavelet collocation method numerically. The presented concept and method was verified by means of different examples, where theoretical results were numerically confirmed. Most computations were performed using MATLAB codes.computations were performed using MATLAB codes. This project can be extending to solve nonlinear volterra-fredholm-hammerstein integral equation with inequality constraints using these two methods.

Key words: Non-linear Volterra-Fredholm-Hammerstein integral equations, Haar Wavelet Collocation Method, Operational Matrix and Leibnitz rule.

1. INTRODUCTION

1.1. Background of the Study

The integral equation is one of the most useful mathematical tools in both pure and applied mathematics. An integral equation is defined as any functional equation in which the unknown function appears under the sign of integration. Integral equations arise in a many branches of science; For example, in potential theory, acoustics, elasticity, fluid mechanics, radioactive transfer, theory of population, etc (Smetanin, 1991). In many instances, the integral equation originates from the conversion of a boundary value problem or an initial value problem associated with a partial or an ordinary differential equation, but many problems lead directly to integral equations cannot be formulated in terms of differential equations.

Several numerical methods for approximating the solution of linear and non-linear integral equations (and especially non-linear integral equations) are known. Integral equations are of many types; here we attempt to indicate some of the main distinguishing features with particular regard to the use and construction of algorithms. In the classical theory of integral equations, one classifies them as between Volterra integral equations, Fredholm integral equations, Singular integral equations and Hammerstein integral equations (parand and Rad, 2012).

Fredholm integral equation is one of the most important integral equations. Integral equations can be viewed as equations which are results of transformation of points in a given vector spaces of integrable functions by the use of certain specific integral operators to points in the same space (Rabbani and Kiasoltani, 2011). If, in particular, one is concerned with function spaces spanned by polynomials for which the kernel of the corresponding transforming integral operator is separable being comprised of polynomial functions only, then several approximate methods of solution of integral equations can be developed. Fredholm integral equation (FIE) is an integral equation in which the limit of integration is fixed.

The FIE arises from the conversion of boundary value problems associated with an ordinary differential equation. The non-linear Fredholm integral equations of the second kind are characterized by fixed limits of integration of the form

$$u(x)=\mathrm{f}(x)+\int_a^b k(x,t)\,\varphi\big(x,u(t)\big)dt \tag{1.1}$$

where $\mathrm{u}(x)$ is the unknown function defined on [a, b] , $f(x), k(x,t)$ are known functions and the function φ is a given continuous function in $[a,b] \times (-\infty,\infty)$ non-linear in u(x).

On the other hand a Volterra Integral equation (VIE) the limit of integration is variable and arises from initial value problems associated with a partial or an ordinary differential equation. The VIE is one of the important subjects of applied mathematics which occurs as a reformulation of ordinary or partial differential equation, or directly appears as a mathematical model of numerous applications in engineering, mathematical physics,economics, biology and etc (Miller, 1967). The non-linear Volterra equations are characterized by at least one variable limit of integration. In the non-linear Volterra integral equations of the second kind, the unknown function u(x) appears inside and outside the integral sign.
The non-linear VIE of the second kind is represented by the form

$$\mathrm{u(x)} = \mathrm{f}(x) + \int_a^x k(x,t)\,\varphi\big(x,u(t)\big)dt \tag{1.2}$$

where $\mathrm{u}(x)$ is the unknown function defined on [a, b] , $f(x), k(x,t)$ are known functions and the function φ is a given continuous function in $[a,b] \times (-\infty,\infty)$ non-linear in $u(x)$.

Singular integral equations have received considerable interest in the mathematics literature, because of their many fields of applications. For example in applied mathematics arise in areas as diverse in the theory of elasticity, viscoelasticity, hydrodynamics and others. Singular integral equation is an integral equation, if the limits of integration are infinite, or the kernel has singularity, in other words the kernel is unbounded within its domain of definition and which has a great role in variety fields for different applications, for example, mathematical physics, engineering, hydromechanics, etc. An approximate methods for multi-dimensional weakly singular integral operators with operator-valued kernels are investigated and a polynomial collocation method also used for finding the numerical solution of a Singular integral equation over the interval by (Saeed and Hassan, 2014). Erdogan et al. (1973) studied Cauchy-type singular integral equation by different numerical techniques. The singular integral equations of the second kind of the form

$$\mathrm{u(x)} = \mathrm{f(x)} + \alpha\int_{-\infty}^{\infty} \mathrm{k(x,t)f(x,u(t))d}t \tag{1.3}$$

where α is a constant parameter f(x) and $k(x,t)$ are given continuous function and $u(x)$ is the unknown to be determine.

Hammerstein integral equation (HIE) of mixed type has been one of the principal tools in various areas of applied mathematics, physics and engineering that is encountered in a variety of applications in many fields including many problems in mathematical physics, the dynamic model of chemical reactor. Some phenomena which appear in many areas of scientific fields such as plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modelled by non-linear integral equations in particular Hammerstein integral equations (Atkinson,1997). Several numerical methods for approximating the solution of Hammerstein integral equations are known. The existence and uniqueness theorems have been proved for equations of the Hammerstein type (Chepanovich, 1984). A non-linear integral equation of Hammerstein type is one of the forms

$$\varphi(\mathrm{x})=\int_a^b k(x,y)\mathrm{f}[\mathrm{y},\varphi(y)]dy \tag{1.4}$$

where $k(x,y)$ is a continuous kernel in L_2 and $f(x,y)$ is a non-linear function of y, continuous in x over the interval $a \leq x \leq b$ and continuous in y over the interval $-\infty \leq y \leq \infty$.

Integral equation which contains Volterra type, Fredholm type and Hammerstein type at the same time are said to be non-linear Volterra-Fredholm-Hammerstein integral equations (VFHIE). Which arise mixed value problem associated with partial or ordinary differential equations. We considered the numerical solution of the non-linear integral equations of the form

$$u(x) = f(x) + \int_a^x k_1(x,t)\varphi_1\big(t,u(t)\big)dt + \int_a^b k_2(x,t)\varphi_2\big(t,u(t)\big)dt,\ x \in [a,b]$$

where $u(x)$ is a solution to be determined in $[a,b], f(x), k_1(x,t), k_2(x,t)$ are known functions and the functions φ_1, φ_2 are given continuous functions in $[a,b] \times (-\infty,\infty)$ non - linear in $u(x)$. Integral equations of type arise in the mathematical modelling of parabolic boundary value problems and in various areas of physics, engineering and biological sciences (Sohrab, 2016). Numerical solution of non-linear integral equation has been the subject of several studies. For the case $\varphi_1\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big) = \mathrm{F}_1(\mathrm{u}(\mathrm{t}))$ and $\varphi_2(\mathrm{t},\mathrm{u}(\mathrm{t})) =$

$F_2(u(t)$,where $F_1(u(t))$ and $F_2(u(t))$ are given continuous functions which are non-linear with respect to u (t).

In this project, we intend to apply HWCM to approximate the solution of non-linear integral equation. The haar wavelet collocation method is a frequently used numerical technique in practical engineering problems because of its easy implementation, particularly for BIE for two-dimensional boundary value problems. Haar wavelet method is pplied for different kind of integral equations, which among Lepik et al.(2007) presented the solution for differential and integral equations. Haar wavelet collocation method is applied to solve system of two dimensional linear and non-linear Fredholm and Volterra integral equations. Using the Jacobi polynomials, two dimensional integral equations reduce to a system of algebraic equations. The main aim is the developing the operational matrices of integration and product for the solving system of two dimensional Fredholm and Volterra integral equations. These matrices together with the haar wavelet collocation method are applied to reduce the solution of these problems to the solution of a system of algebraic equations and the numerical examples illustrate the efficiency and accuracy of this method. The operational matrix of integer integration has been determined for several types of orthogonal polynomials, such as Chebyshev polynomials (Paraskevopoulos, 1983), Legendre polynomials (Paraskevopoulos, 1985), and Laguerre and Hermite (Ahmed et al., 2009). Recently, Singh et al. (2009) derived the Bernstein operational matrix of integration.

1.2. Statement of the Problem

Analytic solutions of the second kind non-linear integral equations are usually difficult. In many cases, it is required to approximate solutions. In this project work, the system of second kind non-linear volterra-fredholm-hammerstein integral equations were solved by using haar wavelet collocation method. Integral equations, depending on the structure of integrals, have different types, for example Fredholm integral equations, Volterra integral equations, Hammerstein integral equation and VFHIE. Integral equation which contains Volterra type, Fredholm type and Hammerstein type at the same time are said to be non-linear Volterra-Fredholm-Hammerstein integral equations (VFHIE). We considered the numerical solution of the non-linear VFH integral equations of the form

$$u(x) = f(x) + \int_{a}^{x} k_1(x,t)\varphi_1\big(t,u(t)\big)dt + \int_{a}^{b} k_2(x,t)\varphi_2\big(t,u(t)\big)dt \quad ,x \in [a,b] \tag{1.7}$$

where $u(x)$ is a solution to be determined in $[a,b], f(x), k_1(x,t), k_2(x,t)$ are known functions and the functions φ_1, φ_2 are given continuous functions in $[a,b] \times (-\infty,\infty)$ non -linear in $u(x)$. Integral equations of type (1.7) arise in the mathematical modelling of parabolic boundary value problems and in various areas of physics, engineering and biological sciences (Sohrab, 2016). Numerical solution of non-integral equation (1.7) has been the subject of several studies. For the case $\varphi_1\big(t,u(t)\big) = F_1(u(t))$ and $\varphi_2(\mathrm{t},\mathrm{u}(\mathrm{t})) = \mathrm{F}_2(\mathrm{u}(\mathrm{t})$,where $F_1(u(t))$ and F_2(u(t)) are given continuous functions which are non-linear with respect to u (t).

1.3. Objectives of the Study

The general objective of this study is to determine numerical solution of non-linear VFHIE by using HWCM. The Specific objectives of this project were:-

- To elaborate haar wavelet collocation method.
- To compute the numerical solutions of non-linear integral equations by using Haar Wavelet Collocation Method.
- To perform error analysis of the proposed method numerically.
- To illustrate the efficiency of the proposed method by solving different non-linear volterra-fredholm-hammerstein integral equations.

2. REVIEW OF RELATED LITERATURE

2.1. Non-Linear Integral Equation

Integral equations are the most useful mathematical tools in both pure and applied mathematics. They have enormous applications in many practical problems. Many initial and boundary value problems associated with (ODE) and (PDE) can be transformed into problems of solving some approximate integral equations. Any functional equation in which the unknown appears under the sign of integration is called an integration equation.Integral equations are encountered in various fields of science and have numerous applications (inelasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynami cs, filtration theory, electrostatics, electrodynamic, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.) (Andrei and Alexander, 2008). Many physical processes and mathematical models are usually governed by the integral equations.

In particular, many initial and boundary value problems can easily be converted to integral equations. Since integral equation has many potential application areas, it has attracted many researchers' attentions from past to today. Integral equations have proved itself as one of the most important branches of mathematics. The theory of integral equations has close contacts with many different areas of mathematics. Foremost among these are differential equations and operator theory. Many problems in the fields of ordinary and partial differential equations can be recast as integral equations (Borzabadi, 2006). Integral equations arise naturally in physics, chemistry, biology, engineering and many physical phenomena (Mohamad, 1985). There are several methods for approximating the solution of linear and non-linear integral equations.

Integral equations have motivated a large amount of research work in recent years. Integral equations find its applications in various fields of mathematics, science and technology has been studied extensively both at the theoretical and practical level. In particular, integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc. In most of the cases, it is difficult to solve them, especially analytically (Hashmi, 2012). Analytical solutions of integral equations, however, either does not exist or are difficult to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations. Therefore variety of analytical and numerical methods like have been used to handle integral equations were developed, haar wavelet collocation method. Non-linearity is one of the interesting topics among the physicists, mathematicians, engineers, etc. Since most physical systems are inherently non-linear in nature. The mathematical modelling of many scientific real world problems occurs non-linearly.

Integral equations of various types and kinds play an important role in many branches of mathematics. Foremost among these are differential equations and theory of operators. Many problems of fracture mechanics, aerodynamics, the theory of porous filtering, antenna problems in electromagnetic theory and others can be formulated as integral equations of the first, second and third kind. The solutions of their applications can be obtained analytically, using the theory developed by (Muskhelishvili, 1949). At the same time the sense of numerical methods takes an important place in solving integral equations. Integral equations have motivated a large amount of research work in recent years.

An integral equation is an equation wherein an unknown function appears under one or more integral signs (Atkinson, 1997). Integral equations occur naturally in many fields of mechanics and mathematical physics. Integral equations also form one of the most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic processes. Many problems which are solved by differential equations methods can be solved more effectively by integral equations. We often convert differential equations into integral equations with boundary conditions or with initial conditions built in. Indeed, a differential equation can be replaced by an integral equation which incorporates its boundary conditions. Each solution of integral equation automatically satisfies these boundary conditions. Analytical solutions of integral equations, however, either does not exist or are difficult to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations.

Therefore variety of analytical and numerical methods has been used to handle integral equations. There are many works on developing and analyzing numerical methods for solving the 1D Volterra-Fredholm integral equations of the second kind. But little work has been done to solve the 2D cases.

In the work of Maleknejad *et al*.(2010) non-linear integral equations are solved numericaly by using HWCM. According to their work operational matrices of integration and the product for alternative Legendre polynomials are first derived. Also, a numerical method for computing approximate solutions of non-linear Volterra-Fredholm–Hammerstein integral equation of the second kind is described. Ordokhani and Razzaghi (2008) applied the rationalized Haar functions to obtain the solutions and similar methods based on operational matrices of Bernstein and Bernoulli polynomials ,respectively proposed by (Bazm,2015) for the numerical solution of (1.7). A composite collocation method based on the hybrid of block-pulse functions and Lagrange polynomials is also proposed by (Marzban et al., 2011).

2.2. Haar Wavelet Collocation Method

HWCM is a widely popular numerical technique in solving integral equations, differential equations, etc. When haar wavelet collocation method is used to solve complicated engineering problems, and it has several disadvantages, that is, low efficiency, illconditioned, etc. Thus, different types of techniques were proposed to improve thecomputational performance of haar wavelet collocation method. The haar wavelet collocation method is transforming a given integral equation into a system of non-linear equations, which has to be solved with some kind of iterative method (Kumar and Sloan, 1987). Several numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm–Hammerstein integral equations, the classical method of successive approximations was introduced in (Ordokhani and Razzaghi, 2008).

The HWCM is a frequently used numerical technique in practical engineering problems because of its easy implementation, particularly for BIE for two-dimensional BVP. A variation of the HWCM was presented in (Lardy, 1987). A HWCM type method was developed in (Kumar and Sloan, 1982). In Brunner, (1992) applied a collocation-type method to non-linear Volterra-Hammerstein integral equations and integrodifferential equations , and discussed its connection with the iterated haar wavelet collocation method. Guoqiang (1993) introduced and discussed the asymptotic error expansion of a haar wavelet collocation-type method for Volterra–Hammerstein integral equations. System of integral equations also appears in many real world problems. So finding approximate solutions for these systems is a challenging problem for many researchers.

In Rabbani (2011), the discrete haar wavelet collocation method was used for solving system of non-linear Volterra-Fredholm-Hammerstein integral equation.

Haar Wavelet Collocation method have several desirable properties. They provide an approximation over the entire integration interval to the solution of the equation, which reveals to be quite useful in a variable-step size implementation: indeed, it is easy to recover the missing past values when the step size is changed, by evaluating the collocation polynomial. Other good properties of haar wavelet collocation methods are their high order of convergence, strong stability properties and flexibility. As a matter of fact, if some information is known on the behaviour of the exact solution, then it is possible to choose the collocation functions in order to better follow such behaviour, so giving rise to mixed collocation Methods.

When a HWCM is applied to an integral equation, several integrals must be computed, thus suitable quadrature rules are needed to complete the discretization, with the introduction of an additional error.Lastly, a reliable error estimation for haar wavelet collocation method for integral equations is still missing: there have been some advances; however considerable work needs to be done. One-step HWCM first appeared in the literature and main results are collected in the monographs (Brunne, 2004).

2.3. Operational Matrix

Yousefi and Behroozifar (2010) derived the OM of Bernstein polynomials and used that OM for numerical solution of non-linear VFHIE integral equation. The advantage of Bernstein operational matrices to that HWCM is simplicity of implementation beside very better answers. The operational matrix of fractional and integer derivatives has been determined for some types of orthogonal polynomials such as flatlet oblique multiwavelets (Darani et al., 2010). The OM of Legendre wavelet is generalized for fractional calculus in order to solve fractional and classical Riccati differential equations.

The OM approach is developed for solving a system of high-order linear Volterra-Fredholm integro-differential equations (Maleknejad et al., 2012). The operational matrix of fractional and integer derivatives has been determined for some types of orthogonal polynomials, such as flatlet oblique multiwavelets (Darani et al., 2010), B-spline cardinal functions (Lakestani et al., 2012), Chebyshev cardinal functions (Dehghan and Lakestani, 2009), Chebyshev polynomials (Doha et al., 2011), and Legendre polynomials (Saadatmandi and Dehghan, 2010).

The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Laguerre series (Hwang and Shi, 1982), Chebyshev polynomials (Paraskevopoulos, 1983), Legendre polynomials (Paraskevopoulos, 1985) and Bessel series (Paraskevopoulos et al., 1990). Recently, Singh et al., (2009) derived the Bernstein operational matrix of integration. Also we refer the interested reader to Saadatmandi et al., (2012) for more research works on the numerical solution of the fractional differential equations.

3. MATERIALS AND METHODS

This project work the method that was used to solve non-linear volterra-fredholm-hammerstein integral equations, and materials used in the study. Sources in the web and libraries was used to collect all the pieces of information about non-linear volterra-fredholm-hammerstein, haar wavelet collocation method.Specifically,

- ✓ Relevant journals and books consulted to gather information about non-linear volterra-fredholm-hammerstein integration equations and haar wavelet collocation method.
- ✓ The collected information analyzed and arranged.
- ✓ Important concepts, definitions, examples and theorems were discussed to make ideas clear.
- ✓ Haar Wavelet Collocation method was used to solve non-linear volterra-fredholm-hammerstein integral equations.
- ✓ Error analysis was done by haar wavelet collocation method numerically.
- ✓ MATLAB was employed to carry out all complicated computational tasks may arise in the solution method.

4. PRELIMINARY

In this chapter, we deal with theoretical concepts, definitions and theorems which are important for the study of HWCM to solve non-linear VFHIE.

4.1. Non-linear Integral Equations

The non-linear VFHIEs arise in the theory of parabolic BVPs, engineering, various mathematical physics, and theory of elasticity (Smetanin, 1991). In recent years, several analytical and numerical methods of this kind of problems have been presented (Linz, 1985); the authors used Taylor series to solve the following non-linear VFHIE:

$$y(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)[y(t)]^p dt + \lambda_2 \int_0^1 k_2(x,t)[y(t)]^q dt \quad p,q \in R. \quad (4.1)$$

where as the Legendre method for a special type was applied in (4.1) for solving the non-linear Volterra-Fredholm-Hammerstein integral equation of the form

$$y(x) = f(x) + \lambda_1 \int_0^x k_1(x,t)[F(y(t))]dt + \lambda_2 \int_0^1 k_2(x,t)G(y(t))dt,$$

Where $f(x)$ and the kernels $k_1(x,t)$ and $k_2(x,t)$ are assumed to be in $L^2(R)$on the interval $0 \le x,t \le 1$. The non-linear Volterra-Fredholm-Hammerstein integral equation is given as follows:

$$y(t) = f(t) + \lambda_1 \int_0^t k_1(t,s)[g_1(s,y(s))]ds + \lambda_2 \int_0^1 k_2(t,s)g_2(s,y(s))ds, 0 \le t,s \le 1$$

In this project, we introduced a method to find the numerical solution of a non-linear VFHIE of the form:

$$\phi(t) = f(t) + \lambda_1 \int_0^t V(t,s,\phi(s))\,ds + \lambda_2 \int_a^b F(t,s,\phi(s))ds, \qquad 0 \le t,s \le 1,$$

(4.2)

where $f(t), V(t,s,\phi(s))$ and $F(t,s,\phi(s))$, are assumed to be in $L^2(R)$ and satisfy the Lipch itz condition

$$\left|K\left(t,s,\phi_1(s)\right) - K\left(t,s,\phi_2(s)\right)\right| \le N(t,s)\left|\phi_1(s) - \phi_2(s)\right|.$$

Let the VFH given in (4.2) be written in the form

$$\phi(t) = f(t) + \lambda \int_0^t K(t,s,\phi(s))\,ds, \quad (4.3)$$

such that

$$K(t,s,\phi(s)) = G(t,s,\phi(s)) + F(t,s,\phi(s)),$$

$$G\big(t,s,\phi(s)\big) = e\big(t,s,\phi(s)\big)V\big(t,s,\phi(s)\big),$$

$$e\big(t,s,\phi(s)\big) = \begin{cases}1 & a < s < t < b\\ 0 & s > t,\end{cases}$$

and the kernel $K\big(t,s,\phi(s)\big) \in C[a,b] \times [a,b]$ satisfies

$$\big|K\big(t,s,\phi(s)\big)\big| \le M, \qquad |f(t)| \le K, \tag{4.4}$$

where M, K are arbitrary constants. It is easy to see that (4.3) can be written as follows: $\phi(t) - f(t) = \int_a^t [\,\phi(s) - f(s)]ds + [\phi(a) - f(a)]$

$$= \int_a^b \delta[\,\phi(s) - f(s)]ds + [\phi(a) - f(a)] \text{ where } \delta = \begin{cases}1, & a < s < t < b\\ 0, & s > t\end{cases}$$

then

$$\int_a^b \delta[\,\phi(s) - f(s)]ds + [\phi(a) - f(a)] = \lambda \int_a^b K\big(t,s,\phi(s)\big)\,ds$$

since

$$\phi(s) - f(s)] = \lambda \int_a^b K\big(a,s,\phi(s)\big)\,dt$$

therefore

$$\int_a^b \delta[\,\phi(s) - f(s)]ds + \lambda \int_a^b [K\big(a,s,\phi(s)\big) - K\big(t,s,\phi(s)\big)]ds = 0$$

Let $G(t) = \int_a^b \{\delta[\,\phi(s) - f(s)] + \lambda \int_a^b [K\big(a,s,\phi(s)\big) - K\big(t,s,\phi(s)\big)]\}ds = 0$

that is, if

$$|G(t)| = 0. \tag{4.5}$$

By integrating (4.4) we have

$$\int_a^b |G(t)|dt = 0 \tag{4.6}$$

On the other hand, one can define the following equality:

$$F(t,s,\phi(t),u(t)) = \delta[\phi(s) - f(s)] + \lambda[K\big(a,s,\phi(s)\big) - K\big(t,s,\phi(s)\big)] \tag{4.7}$$

This will lead us to the following inequality:

$$\int_a^b |G(t)|dt \le \int \int_a^b |F(t,s,\phi(t),u(t))|\,dsdt \tag{4.8}$$

where

$$\phi(s) = u(s), \qquad s \in [a,b] \tag{4.9}$$

With the boundary conditions

$$\phi(a) = f(a) + \int_a^b K\big(a,s,\phi(s)\big)\,ds,$$

$$\phi(b) = f(b) + \int_a^b K\big(b, s, \phi(s)\big)\, ds \tag{4.10}$$

$\phi(a)$ and $\phi(b)$ are defined in (4.10) where $\Omega = [a, b] \times [a, b]$. The existence and uniqueness of (4.3) will be considered in the next section by using the successive approximation method.

The solution $\phi(t)$ of (4.3) can be approximated successively as follows:

$$\phi_1(t) = \int_a^b K\big(b, s, \phi_0(t)\big)\, ds = f(t). \tag{4.11}$$

Thus, we obtain sequence of functions $\phi_0(t), \phi_1(t), \dots, \phi_n(t)$ such that

$$\phi_n(t) - \lambda \int_a^b K\big(t, s, \phi_{n-1}(s)\big) ds = f(s), \quad n \geq 1 \tag{4.12}$$

With $\phi_0(t) = f(t)$. It is convenient to introduce

$$\psi_n(t) = \phi_n(t) - \phi_{n-1}(t), \qquad n \geq 1 \tag{4.13}$$

With $\psi_n(t) = f(t)$.

Subtracting from (4.12) the same equation with replacing n by $n - 1$, we get

$$\phi_n(t) - \phi_{n-1}(t) = \lambda \int_a^b K\big(t, s, \phi_{n-1}(s)\big) ds - \lambda \int_a^b K\big(t, s, \phi_{n-2}(s)\big) ds \tag{4.14}$$

Using (4.13), we have

$$\psi_n(t) = \lambda \int_a^b K\big(t, s, \psi_{n-1}(s)\big) ds \tag{4.15}$$

Also, from (4.13), we deduce that

$$\phi_n(t) = \sum_{i=0}^{n} \psi_i(t). \tag{4.16}$$

The existence and uniqueness of the solution can be followed.

Theorem 4.1. If the kernel $K\big(t, s, \phi(s)\big)$ and the function $f(t)$ are continuous and satisfy condition (4.11) in $a < s < t < b$, then the integral equation (4.3) possesses a unique continuous solution.

Proof. From (4.15), we get

$$\begin{aligned} \left|\psi_n(t)\right| &= \left|\lambda \int_a^b K\big(t, s, \psi_{n-1}(s)\big) ds\right| \\ &\leq \lambda \left|K\Big(t, s, \psi_{n-1}(s)\Big)\right| \int_a^b ds \\ &\leq \lambda(a - b)M \end{aligned} \tag{4.17}$$

We now show that this $\phi(t)$ satisfies (4.3).

The series (4.16) is uniformly convergent since the term $\psi_i(t)$ is dominated by $\lambda(a, b)M$.

Then,

$$\begin{aligned} \lambda \int_a^b K(t, s, \textstyle\sum_{i=0}^{\infty} \psi_i(s)) ds &= \textstyle\sum_{i=0}^{\infty} \lambda \int_a^b K\big(t, s, \psi_i(s)\big) ds \\ &= \textstyle\sum_{i+1}^{\infty} \psi_i(t) \end{aligned}$$

$$=\sum_{i+1}^{\infty}\psi_i(t)+\psi_0(t)-\psi_0(t) \tag{4.18}$$

Hence, we have

$$\lambda\int_a^b K(t,s,\sum_{i=o}^{\infty}\psi(s))ds=\sum_{i=0}^{\infty}\psi(s)-f(t) \tag{4.19}$$

This proves that $\phi(t)$ defined in (4.16) satisfies (4.3) since each of the $\psi_i(t)$ is clearly Continuous; therefore $\phi(t)$ is continuous, where it is the limit of a uniformly convergent sequence of continuous functions. To show that $\phi(t)$ is a unique continuous solution, suppose that there exists another continuous solution $\tilde{\phi}(t)$ of (4.3), then

$$\tilde{\phi}(t)-\lambda\int_a^b K\left(t,s,\tilde{\phi}(s)\right)ds=f(t). \tag{4.20}$$

Subtracting (4.20) from (4.3), we get

$$\phi(t)-\tilde{\phi}(t)=\lambda\int_a^b K\big(t,s,\{\phi(t)-\tilde{\phi}(t)\}\big)ds. \tag{4.21}$$

Since $\phi(t)$ and $\tilde{\phi}(t)$ are both continuous, there exists a constant *B* such that

$$||\phi(t)-\tilde{\phi}(t)|\leq B. \tag{4.22}$$

By using the condition of (4.4), the inequality (4.22) becomes

$$|\phi(t)-\tilde{\phi}(t)|\leq\lambda(b-a)MB \tag{4.23}$$

For the large enough n, the right-hand side is arbitrary small, then

$$\phi(t)=\tilde{\phi}(t) \tag{4.24}$$

This completes the proof.

4.1.1. Classification of Non-Linear Integral Equations

Based on the limit of non-linear integration or type of kernel the integral equation can be classified as

1) Volterra integral equations
2) Fredholm integral equations
3) Singular integral equations
4) Hammerstein Integral Equations

Not only limits of integration are used to characterize integral equation but also type of kernel and appearing derivatives of unknown function determine types of integral equation.

4.1.1.1. Non-linear Volterra integral equations

It is well known that linear and non-linear Volterra integral equations arise in many scientific fields such as the population dynamics spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by (du Bois-Reymond, 1888). However, the name Volterra integral equation was first coined by (Lalesco, 1908). It is our goal in this chapter to study the non-linear Volterra integral equations of the second kind. The non-linear Volterra equations are characterized by at least one variable limit of integration. In the non-linear Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The non-linear Volterra integral equation of the second kind is represented by the form:

$$u(x) = f(x) + \int_a^x k(x,t)F\big(u(t)\big)dt$$

However, the non-linear Volterra integral equations of the first kind contain the non-linear function $F\big(u(x)\big)$ inside the integral sign. The non-linear Volterra integral equation of the first kind is expressed in the form

$$f(x) = \int_a^x k(x,t)F\big(u(t)\big)dt$$

For these two kinds of equations, the kernel $k(x,t)$ and the function $f(x)$ are given real-valued functions, and $\mathrm{F}\big(\mathrm{u(x)}\big)$ is a non-linear function of $\mathrm{u(x)}$ such as $\mathrm{u}^3(\mathrm{x})$,$\sin\big(u(x)\big)$, and $\mathrm{e}^{\mathrm{u(x)}}$.

We rewrite the non-linear Volterra integral equation of the second kind by:

$$\mathrm{u(x)} = \mathrm{f(x)} + \int_{\mathrm{a}}^{\mathrm{x}} \mathrm{G(x, t, u(t))dt}$$

The specific conditions under which a solution exists for the non-linear Volterra integral equation are:

(i) The function $\mathrm{f(x)}$ is integrable and bounded in $\mathrm{a} \leq \mathrm{x} \leq \mathrm{b}$.

(ii) The function $\mathrm{f(x)}$ must satisfy the Lipschitz condition in the interval (a,b). This means that

$$|\mathrm{f(x)} - \mathrm{f(y)}| < k|x-y|.$$

(iii) The function $\mathrm{G(x, t, u(t))}$ is integrable and bounded $|\mathrm{G(x, t, u(t))}| < k$ in $a \leq x, t \leq b$.

(iv) The function $\mathrm{G(x, t, u(t)}$ must satisfy the Lipschitz condition

$$|\mathrm{G(x, t, z)} - \mathrm{G(x, t, z')}| < M|z-z'|$$

4.1.1.2. Non-linear Fredholm integral equations

The non-linear Fredholm integral equations of the second kind are characterized by fixed limits of integration of the form

$$u(x) = f(x) + \int_a^b k(x,t)F\big(u(t)\big)dt$$

where the unknown function $u(x)$ occurs inside and outside the integral sign, λ is a parameter, and a and b are constants. For this type of equations, the kernel $k(x,t)$ and the function $f(x)$ are given real-valued functions, and $F\big(u(t)\big)$ is a non-linear function of $u(x)$ such as $\mathrm{u}^2(x), \sin(u(x))$, and $\mathrm{e}^{\mathrm{u(x)}}$. We use degenerate or separable kernels. A degenerate or a separable kernel is a function that can be expressed as the sum of product of two functions each depends only on one variable. Such a kernel can be expressed in the form

$$K(x,t) = \sum_{i=1}^{n} g_i(\mathrm{x})\, f_i(\mathrm{t}).$$

We rewrite the non-linear Fredholm integral equation of the second kind by

$$\mathrm{u(x) = f(x) +}\ \lambda \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{G(x,t,u(t))dt}$$

The specific conditions under which a solution exists for the non-linear Fredholm integral equation are:

(i) The function f(x) is bounded, $|f(x)| < R$, in $a \le x \le b$.

(ii) The function $\mathrm{G(x,t,u(t))}$ is integrable and bounded where

$$|\mathrm{G(x,t,u(t))}| < K, \text{ in } a \le x,t \le b.$$

(iii) The function $\mathrm{G(x,t,u(t))}$ satisfies the Lipschitz condition

$$|\mathrm{G(x,t,z) - G(x,t,z')}| < M|z - z'|.$$

Using the successive approximations method, it is proved in that the series obtained by this method converges uniformly for all values of λ for

$$\lambda < \frac{1}{k(b-a)},$$

where k is the larger of the two numbers $K\left(1 + \frac{R}{|\lambda\,|K(b-a)}\right)$ and M.

Consider the Fredholm integral equations given in (4.12).We first approximate $y(x)$ as

$$y(x) = \mathrm{C}^{\mathrm{T}}\psi(\mathrm{x})$$

$$f(x) = \mathrm{d}^{\mathrm{T}}\psi(\mathrm{x})$$

and

$$K(x) = \psi(\mathrm{x})^{\mathrm{T}}\mathrm{k}\psi(\mathrm{t})$$

where C, and $\psi(x)$ are defined similarly also K is $2^k(2M+1)\times 2^k(2M+1)$ matrices where the elements of K calculated as follows:

$$\int_0^1\int_0^1 \psi_{ni}(x)\,\psi_{lj}(t)K(x,t)dt\,dx,$$

where

$n=1,\dots,2^k$,$i=-M,\dots,M$,$l=1,\dots,2^k$,$j=-M,\dots,M$.

Then we have

$$C^T\psi(x)=d^T\psi(x)+\lambda\int_0^1\psi(x)^T K\psi(t)\psi(t)^T Cdt \tag{4.25}$$

Thus with the orthonormality became we have

$$\psi(x)^T C=\psi(x)^T d+\lambda\psi(x)^T KC \tag{4.26}$$

(4.25) is a linear systems interms of C and the answer is

$C=(I-K)^{-1}d$ where I is identity matrix.

Example 4.2. Consider the non-linear Fredholm integral equation by

$$Y(x)=\frac{1}{4}x+\cos(2\pi x)-\int_0^1(xt)\,y^2(t)$$

Let $y(x)=C^T\psi(x)$,$\frac{1}{4}xt+\cos(2\pi x)=d^T\psi(x)$,$x=h^T\psi(x)$ and $t=h^T\psi(t)$. Thus (4.1) becomes

$$C^T\psi(x)=d^T\psi(x)-\int_0^1\psi(x)^T hh^T\psi(t)C^T\psi(t)\psi(t)^T Cdt$$

Using product operational matrix, we have

$$\psi^T(x)C=\psi^T(x)d-\psi^T(x)h\int_0^1 h^T\psi(t)\,\psi^T(t)\hat{C}Cdt$$

and by orthonormality we have

$$\psi^T(x)C=\psi^T(x)d-\psi^T(x)hh^T\hat{C}C$$

and then

$$C-d+hh^T\hat{C}C=0$$

The unknown vectors C obtain with solution of the above algebraic system. For $M=1$ and $K=0$ we have

$C_{1(-1)}=0.5$, $C_{10}=0$, $C_{11}=0.5$

and then $y(x)=\cos(2\pi x)$ which is exact solution

4.1.1.3. Non-linear singular integral equations

A singular integral equation is an equation in which one or both limits of integration are infinite or the kernel becomes infinite at one or more points within the range of integration.

VIE of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt,$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)dt,$$

are called singular if:

1. one of the limits of integration $g(x), h(x)$ or both are infinite, or
2. if the kernel $K(x,t)$ becomes infinite at one or more points at the range of integration.

In a similar manner, non-linear Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)F(u(t))dt,$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)F(u(t))dt,$$

where $F(u(t))$ is a non-linear function of $u(t)$, are called singular if:

1. one of the limits of integration $g(x), h(x)$ or both are infinite, or
2. If the kernel $K(x,t)$ becomes infinite at one or more points at the range of integ ration.

Examples of the second style are the non-linear Abel's integral equation, generalized non-linear Abel's integral equation, and the non-linear weakly-singular integral equations given by

$$f(x) = \int_0^x \frac{1}{\sqrt{(x-t)}} u^2(t)dt,$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u^3(t)dt,\ 0 < \alpha < 1,$$

and $$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u^2(t)dt,\ 0 < \alpha < 1,$$

respectively. It is clear that the kernel in each equation becomes infinite at the upper limit t = x.

Definition 4.1: the kernel of the integral equation is of the form $\mathrm{K(x,t)} = \frac{\mathrm{H(x,t)}}{|\mathrm{x-t}|^{\alpha}}$ or $\mathrm{K(x,t)} = \mathrm{H(x,t)ln|x-t|}$, where $\mathrm{H(x,t)}$ is bounded that is several times continuously differentiable $\mathrm{a} \leq \mathrm{x} \leq \mathrm{b}$ and $\mathrm{a} \leq \mathrm{t} \leq \mathrm{b}$ with $\mathrm{H(x,t)} \neq 0$ and $\alpha \in (0,1)$ the integral equation is said to be Weakly Singular integral equation.

Definition 4.2: The kernel is of the form $K(x,t) = \frac{H(x,t)}{(x-t)^2}$ and $H(x,t)$ is differentiable function the integral equation is said to be Strongly Singular integral equation.

4.1.1.4. Non-linear Hammerstein Integral Equations

Consider the numerical solution of the non-linear integral equations of Hammerstein type

$$Y(t) = f(t) + \int_0^1 \mathrm{k(t,s)g(s,y(t))ds}\,, \qquad \mathrm{t\epsilon[0,1]} \tag{4.27}$$

where f, k and g are given functions and y is the unknown. There has been much interest in this problem since Hammerstein integral equations, which came from the electro-magnetic fluid dynamics, yields strong physical background. Moreover, the Fredholm integral equations of second kind are the special case of the Hammerstein integral equations. In this approach some iterative method (Aghazadeh, 2004) is used for solving the corresponding system of non-linear equations and definite integrals need to be evaluated at each step of the iteration. For solving (4.16) in Kumar (1987) collocation-type method for (4.16) was introduced in which the collocation method is applied to an equivalent equation for Hammerstein integral equations, such that we suppose that

$$\mathrm{z}(t) = g(t, y(t)),\ t\epsilon[0,1] \tag{4.28}$$

with substituting (4.17) into (4.16) we reach at

$$Y(t) = f(t) + \int_0^1 \mathrm{k(t,s)z(s)ds}\ , \quad t\epsilon[0,1] \tag{4.29}$$

and conclude that the new unknown $z(t)$ satisfies the non-linear integral equation

$$\mathrm{z(t)} = \mathrm{g}\left(t, f(t) + \int_0^1 \mathrm{k(t,s)z(s)ds}\right),\ t\epsilon[0,1] \tag{4.30}$$

Wang and Lin (1998) developed a degenerate kernel scheme for the Hammerstein integral equations. In this paper we consider general kernels including non-degenerate kernels.

4.1.2. Linearity of Integral Equation

There are two kinds of integral equations according to linearity and this depends on the unknown function under the integral sign.

1. Linear integral equations

They are of the form

$$f(x) = g(x) + \lambda \int_0^t G\,(x, y)f(y)dy,$$

where only linear operations are performed upon the unknown function inside the integral sign, that is the exponent of the unknown inside the integral sign is one, for example

$$f(x) = \frac{3}{2}x - \frac{1}{3} + \int_0^1 (x - y)\, f(y)dy.$$

here the unknown function f appears in the linear form.

2. Non-linear integral equations

They are of the form

$$f(x) = g(x) + \lambda \int_{v(x)}^{u(x)} G\,(x, yf(y))dy,$$

the unknown function f under the integral sign has exponent other than one, or the equation contains non-linear functions of, such as e^{f} , $\sin hf$, $\ln(1+f)$ for example

$$f(x) = 1 + \int_0^x (1 + x - y)\, f^4(y)dy.$$

4.1.3. The Connection between Differential and Integral Equation

An initial or BVPs can be transferred to equivalence integral equations.

Assume that we want to solve the initial value problem

$$\begin{cases} u''(x) + q(x)u(x) = f(x), x > a \\ \qquad u(a) = u_0\,, u'(a) = u_1 \end{cases}.$$

We integrate the equation from a to x and get

$$\int_a^x (u''(y) + q(y)u(y))\, dy = \int_a^x f(y)dy$$

$$u'(x) - u_1 = \int_a^x [f(y) - q(y)u(y)]dy,$$

and integration yields

$$\int_a^x (u'(s) - u_1)ds = \int_a^x \int_a^x [f(y) - q(y)u(y)]\, dyds$$

We get

$$u(x) - u_0 = u_1(x-a) + \int_a^x [f(y) - q(y)u(y)](x-a)dy$$

This can be written as.

$$u(x) = u_0 + u_1(x-a) + \int_a^x f(y)(x-y)dy + \int_a^x q(y)(y-a)u(y)dy$$

$$u(x) = F(x) + \int_a^x k(x,y)u(y)dy\,,$$

Where $F(x) = u_0 + u_1(x-a) + \int_a^x f(y)(x-y)dy$

$$k(x,y) = q(y)(y-x)$$

Lemma 4.1.4. (Leibniz integral rule)

In calculus Leibniz rule for differentiation under the integral sign named after Gottfried Leibniz state that an integral of the form $\int_{y_0}^{y_1} f(x,y)dy$ for x in (x_0, x_1) the derivative of this integral is expressible as

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b(t),t)b^{'}(t) - f(a(t),t)a^{'}(t)$$

where the partial derivative indicates that inside the integral, only the variation of $f(x,t)$ with t is considered in taking the derivative.

Proof: Let,$G(t,a,b) = \int_a^b f(x,t)dx$,where $a(t) = a$, $b(t) = b$.The chain rule now gives

$$\frac{d}{dt}G = G^{'}{}_t(t,a,b) + G^{'}{}_a(t,a,b)a^{'}(t) + G^{'}{}_b(t,a,b)b^{'}(t)$$

$$= \int_a^b f^{'}{}_t(x,t)dx - f(a(t),t)a^{'}(t) + f(b(t),t)b^{'}(t)$$

but $G = \int_a^b f(x,t)dx$,

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_a^b f'_t(x,t)dx - f(a(t),t)a'(t) + f(b(t),t)b'(t)$$

4.2. Errors

Approximation is a central concept in almost all the uses of mathematics. The main purpose of numerical analysis and scientific computing is to develop efficient and accurate methods to compute approximations to quantities that are difficult or impossible to obtain by analytic means. Numerical results are affected by many types of errors. Some of them are:

Inherent error: Errors which are already present in the statement of the problem before its solution.

Rounding error: It is errors that arise from the process of rounding off the numbers during computation. It occurs whenever an irrational number, for example, is shortened ("rounded off") to a fixed number of digits, or when a decimal fraction is converted to the binary form used in the computer. The limitation of floating-point numbers in a computer leads at times to a loss of information that, depending on the context, may or may not be important.

Truncation error: These are errors committed when a limiting process is truncated (broken off) before one has come to the limiting value. A truncation error occurs, for example, when an infinite series is broken off after a finite number of terms, or when a derivative is approximated with a difference quotient (although in this case the term **discretization error** is better). Another example is when a nonlinear function is approximated with a linear function. We use some examples to make the idea more precise. Let we expand $Y(x_{n+1})$ by using Taylor series about x_n as

$$Y(x_{n+1}) = Y(x_n) + Y^{'}(x_n) + \frac{h^2}{2}Y^{''}(\varphi_n), \qquad x_n < \varphi_n < x_{n+1}.$$

To obtain the Euler method, we drop the error term. The dropped term $\frac{h^2}{2}Y^{''}(\varphi_n)$ is called the **truncation error** atx_{n+1}.

Absolute, Relative and Percentage errors: If x is the true value of a quantity and x_a is its approximate value, then the absolute error defined as $E_a = |x - x_a|$. The relative error $E_r$is defined by $E_r = \left|\frac{x-x_a}{x}\right|$,and the percentage error E_p is defined by

$$E_p = 100E_r = 100\left|\frac{x-x_a}{x}\right|.$$

For example, let $x = e = 2.7182818$ ... and $x_a = \frac{19}{7} = 2.7142857$ ... then, $E_a = |x - x_a| = 0.003996$ ..., $E_r = \left|\frac{x-x_a}{x}\right| = 0.00147..$, and $E_p = 100E_r = 0.147$. In this project work we used only the absolute error, but the others were to describing only.

5. NUMERICAL SOLUTION FOR NON-LINEAR VOLTERRA-FREDHOLM-HAMMERSTEIN INTEGRAL EQUATIONS BY USING HAAR WAVELET COLLOCATION METHOD

5.1. Haar Wavelet Collocation Method

The HWCM is employed for non-linear VFHIE that given by the general form

$$\mathrm{y(x)} = \mathrm{f(x)} + \lambda_1 \int_0^{\mathrm{x}} \mathrm{k}_1\,(\mathrm{x,t})\mathrm{G}_1\big(\mathrm{t,y(t)}\big)\mathrm{dt} + \lambda_2 \int_0^1 \mathrm{k}_2\,(\mathrm{x,t})\mathrm{G}_2\big(\mathrm{t,y(t)}\big)\mathrm{dt},\ 0 \le x, t \le 1 \quad (5.1)$$

where $\lambda_1, \lambda_2$are constants and f (x) and kernels $K_1(x,t)$ and $K_2(x,t)$, are given functions on the interval $0 \le x, t \le 1$. In (5.1) $\mathrm{G}_1\big(\mathrm{t,y(t)}\big)$ and $\mathrm{G}_2\big(\mathrm{t,y(t)}\big)$ are given continuous functions which are non-linear with respect to $\mathrm{y(t)}$ and t. These types of equations arise in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc (Maleknejad and Hashemizadeh, 2012).

The numerical solvability of (5.1) and other related equations has been pursued by several authors. For Fredholm-Hammerstein integral equations, the classical method of successive approximations was introduced in (Tricomi, 1982) Lardy in (1992) applied the variation of the haar wavelet collocation method for (5.1). In Brunner (1992), Brunner applied a collocation-type method to non-linear Volterra-Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated haar wavelet collocation method. Marzban et al. solved the non-linear VFHIE by using the Hybrid of block-pulse functions (Marzban, 2011). In Yousefi and Razzaghi (2005) a numerical procedure based on Legendre wavelet approximations was presented for solving a class of the on-linear Volterra-Fredholm integral equations. For more references about numerical method for Volterra-Hammerstein integral equations (Parand, 2012). The idea of the haar wavelet collocation method for ODE with some supplementary conditions is first given by (Liu, 1986).In (Gouyandeh, 2016) this method is applied successfully to find the numerical solution of eigenvalue problems and in Wai Wai, (2004) investigated the haar wavelet collocation method in detail. Recently it is provided an efficient numerical approach for multi-order fractional differential equations based on the haar wavelet collocation method

(Allahviranloo, 2015). The purpose of this project is to solve the non-linear VFHIE using the HWCM.To achieve this purpose, the non-linear VFHIE are transformed into a system of non-linear algebraic equations in matrix form, thus by solving this system unknown coefficients are obtained.

5.2. Haar Wavelet

The scaling function $h_1(x)$ for the family of the Haar wavelet is defined as

$$h_1(\mathrm{x}) = \begin{cases} 1, \ \text{for } \ \mathrm{x} \in [0,1) \\ 0, \ \text{elsewhere} \end{cases} \tag{5.2}$$

The Haar Wavelet family for $x \in [0,1)$ is defined as,

$$h_i(\mathrm{x}) = \begin{cases} 1, \ for\ x \in [\alpha, \beta) \\ -1, \ for\ x \in [\alpha, \gamma), \ i = 2,3,4,\cdots \\ 0, \ elsewhere \end{cases} \tag{5.3}$$

where $\alpha = \frac{\mathrm{k}}{\mathrm{m}}; \beta = \frac{\mathrm{k}+0.5}{\mathrm{m}}; \ \gamma = \frac{\mathrm{k}+1}{\mathrm{m}}$

where $\mathrm{m} = 2^{\mathrm{l}}$, $\mathrm{l} = 0,1,2,3,\cdots,\mathrm{J}$,J is the level of resolution; and $k = 0,1,2,\cdots,\mathrm{m}-1$is the translation parameter. Maximum level of resolution is J. The index i in (5.3) is calculating using $i = m + k + 1$. In case of minimal values $\mathrm{m} = 1, k = 0$ then $i = 2$.The maximal value of i is $\mathrm{N} = 2^{\mathrm{J}+1}$.

Let us define the collocation points $x_j = \frac{j-0.5}{N}, j = 1,2,\cdots,\mathrm{N}$,Haar coefficient matrix

$\mathrm{H}(\mathrm{i},\mathrm{j}) = \mathrm{h}_{\mathrm{i}}(\mathrm{x}_{\mathrm{j}})$ which has the dimension N × N.

For instance, $\mathrm{J} = 3$, $\mathrm{N} = 16$, then we have

H(16,16)

$$
= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
$$

Any function $f(x)$ which is square integrable in the interval $(0,1)$ can be expressed as an infinite sum of Haar wavelets as,

$$\mathrm{f(x)} = \sum_{\mathrm{i}=1}^{\infty} \mathrm{a_i h_i(x)} \tag{5.4}$$

The above series terminates at finite terms if f(x) is piecewise constant or it can be approximated as piecewise constant during each subinterval. Given a function $f(x) \in L^2(\mathcal{R})$ a multi-resolution analysis (MRA) of $L^2(\mathcal{R})$ produces a sequence of subspaces $V_j, V_{j+1}, \cdots$ such that the projections of $f(x)$ onto these spaces gives finer approximation of the function $f(x)$ as $j \to \infty$.

5.2.1. Operational Matrix of Haar Wavelet

The non-linear VFHIE that given by the general form

$$\mathrm{u}(x) = \mathrm{f(x)} + \lambda_1 \int_0^x k_1(x,s)\varphi_1\big(t,u(s)\big)ds + \lambda_2 \int_0^1 k_2(x,s)\varphi_2\big(s,u(s)\big)ds \,, 0 \leq x,s < 1$$

(5.5)

where the parameters λ_1 , λ_2 and functions $f(x)$, $k_1(x,s)$,$k_2(x,s)$,$\varphi_1\big(s,u(s)\big)$ and $\varphi_2\big(\mathrm{s},\mathrm{u(s)}\big)$ are known and $\mathrm{u}(x)$ is an unknown function. The non-linear VFHIE (5.5) arises in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, communication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics, etc.

Consider the non-linear VFHIE (5.5). For implementing the operational matrices on non-linear Volterra-Fredholm-Hammerstein integral equations first we find the collocation approximation to the functions $z_1(x)$ and $z_2(x)$ defined by:

$$z_1(x) = \varphi_1\big(x, u(x)\big),\ z_2(x) = \varphi_2\big(x, u(x)\big) \qquad (5.6)$$

Now by using (5.6) observe that

$$z_1(x) = \varphi_1\left(x, \lambda_1 \int_0^x k_1(x,s)\varphi_1\big(t, u(s)\big)ds + \lambda_2 \int_0^1 k_2(x,s)\varphi_2\big(s, u(s)\big)ds + f(x)\right)$$

$$z_2(x) = \varphi_2\left(x, \lambda_1 \int_0^x k_1(x,s)\varphi_1\big(t, u(s)\big)ds + \lambda_2 \int_0^1 k_2(x,s)\varphi_2\big(s, u(s)\big)ds + f(x)\right) \qquad (5.7)$$

The approximating functions to $z_1(x)$, $z_2(x)$,$k_1(x,s)$ and $k_2(x,s)$ by Bernstein polynomials can be frame by:

$$z_1(x) = z_1{}^T\ \phi(x), z_2(x) = z_2{}^T\ \phi(x),$$

$$k_1(x,s) = \phi^T(x) k_1 \phi(s), k_2(x,s) = \phi^T(x) k_2 \phi(s) \qquad (5.8)$$

which z_1, z_2 and k_1, k_2 are defined with (5.6) and (5.7), respectively. By use to (5.8) we can write the Volterra part of (5.5) as

$$\begin{aligned}\int_0^x k_1(x,s) z_1(s)ds &\simeq \int_0^x \phi^T(x) k_1 \phi(s)\phi(s)^T z_1 ds\\ &= \phi^T(x) k_1 \int_0^x \Pi(s) z_1 ds = \phi^T(x) k_1 \widehat{z_1}^T \int_0^x \phi(s)ds\\ &= \phi^T(x) k_1 \widehat{z_1}^T P \phi(s) \qquad (5.9)\end{aligned}$$

and by use of (5.8) and we have the Fredholm part of (5.5) as follows

$$\begin{aligned}\int_0^1 k_2(x,s) z_2(s)ds &\simeq \int_0^1 \phi^T(x) k_2 \phi(s)\phi(s)^T z_2 ds\\ &= \phi^T(x) k_2 \int_0^1 \phi(s)\phi(s)^T ds\, z_2\\ &= \phi(s)^T k_2 D z_2 \qquad (5.10)\end{aligned}$$

After substituting the approximate (5.8), (5.9) in (5.7) we get

$$\begin{cases} z_1{}^T\phi(x) = \varphi_1\left(x, \lambda_1 \phi^T(x) k_1 k_1 \hat{z}_1{}^T \mathbf{P}\phi(x) + \lambda_2 \phi^T(x) k_2 \mathbf{D} z_2 + f(x)\right)\\ z_2{}^T\phi(x) = \varphi_2\left(x, \lambda_1 \phi^T(x) k_1 k_1 \hat{z}_1{}^T \mathbf{P}\phi(x) + \lambda_2 \phi^T(x) k_2 \mathbf{D} z_2 + f(x)\right)\end{cases} \qquad (5.11)$$

Now we collocate (5.11) in $n + 1$ Newton–Cotes nodes as

$$x_p = \frac{2p-1}{2(n+1)}, p = 1,2, \dots n + 1 \qquad (5.12)$$

then we have (5.11) as

$$\begin{cases} {z_1}^{\mathrm{T}}\phi(\mathrm{x_p}) = \varphi_1\left(\mathrm{x_p}, \lambda_1\phi^{\mathrm{T}}(\mathrm{x_p})\mathrm{k_1 k_1}{\hat{\mathrm{z}}_1}^{\mathrm{T}}\mathbf{P}\phi(\mathrm{x_p}) + \lambda_2\phi^{\mathrm{T}}(\mathrm{x_p})k_2\mathbf{D}z_2 + \mathrm{f}(\mathrm{x_p})\right) \\ {z_2}^{\mathrm{T}}\phi(\mathrm{x_p}) = \varphi_2\left(\mathrm{x_p}, \lambda_1\phi^{\mathrm{T}}(\mathrm{x_p})\mathrm{k_1 k_1}{\hat{\mathrm{z}}_1}^{\mathrm{T}}\mathbf{P}\phi(\mathrm{x_p}) + \lambda_2\phi^{\mathrm{T}}(\mathrm{x_p})k_2\mathbf{D}z_2 + \mathrm{f}(\mathrm{x_p})\right) \\ \qquad\qquad\qquad p = 1,2,\ldots n+1 \end{cases} \tag{5.13}$$

After solving non-linear system (5.13) we get z_1 and z_2.Then we have the approximate solution of (5.14) as follows:

$$u_n = \lambda_1\phi^{\mathrm{T}}(x)\mathrm{k_1}{\hat{\mathrm{z}}_1}^{\mathrm{T}}\mathbf{P}\phi(\mathrm{x}) + \lambda_2\phi^{\mathrm{T}}(\mathrm{x})k_2\mathbf{D}z_2 + \mathrm{f}(\mathrm{x}). \tag{5.14}$$

The operational matrix P which is an N square matrix is defined by

$$\mathrm{p_{1,i}(x)} = \int_0^{\mathrm{x}} \mathrm{h_i(t)dt} \tag{5.15}$$

often, we need the integrals

$$\mathrm{p_{r,i}(x)} = \int_{\mathrm{A}}^{\mathrm{X}}\int_{\mathrm{A}}^{\mathrm{X}}\cdots\int_{\mathrm{A}}^{\mathrm{x}} \mathrm{h_i(t)dt^r} = \frac{1}{(\mathrm{r}-1)!}\int_{\mathrm{A}}^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\mathrm{r}-1}\,\mathrm{h_i(t)dt},\ r = 1,2,\cdots,n$$

and $i = 1,2,\cdots,N$ (5.16)

For $r = 1$ corresponds to the function $\mathrm{p_{1,i}(x)}$, with the help of (5.3) these integrals can be calculated analytically; we get,

$$\mathrm{p_{1,i}(x)} = \begin{cases} x-a,\ for\ x \in [\alpha,\beta) \\ \gamma - x,\ for\ x \in [\beta,\gamma) \\ \quad 0,\ Otherwise \end{cases} \tag{5.18}$$

$$\mathrm{p_{r,i}(x)} == \begin{cases} \frac{1}{2}(x-a)^2,\ for\ x \in [\alpha,\beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2,\ for\ x \in [\beta,\gamma) \\ \frac{1}{4m^2}\ ,for\ \ x \in [\gamma, 1) \\ \quad 0,\ Otherwise \end{cases} \tag{5.19}$$

In general, the operational matrix of integration of r^{th} order is given as

$$\mathrm{p}_{2,\mathrm{i}}(\mathrm{x}) = \begin{cases} \frac{1}{r!}(x-a)^r, \ for\ x \in [\alpha, \beta) \\ \frac{1}{r!}\{(x-a)^r - 2(x-\beta)^r\}, \ for\ x \in [\beta, \gamma) \\ \frac{1}{r!}\{(x-a)^r - 2(x-\beta)^r + (x-\gamma)^r\}\ , for\ \ x \in [\gamma, 1) \\ 0,\ Otherwise \end{cases} \tag{5.20}$$

For instance,$J = 3$, $N = 16$, then we have

$\mathrm{P}_{1,\mathrm{i}}(16, 16)$

$$= \frac{1}{32}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 & 21 & 23 & 25 & 27 & 29 & 31 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and $\mathrm{P}_{2,\mathrm{i}}(16,16) =$

$$\frac{1}{2048}\begin{bmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 289 & 361 & 441 & 529 & 625 & 729 & 841 & 961 \\ 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 & 289 & 343 & 391 & 431 & 463 & 487 & 503 & 511 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 & 128 & 128 & 128 & 128 & 128 & 128 & 128 & 128 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Haar wavelet collocation method (HWCM) based on Leibnitz rule for the numerical solution of non-linear Volterra-Fredholm-Hammerstein integral equation of the form,

$$\mathrm{u(x) = f(x) + \int_0^x k_1\,(x,t)F\big(t,u(t)\big)dt + \int_0^1 k_2\,(x,t)G(t,u(t)),} \tag{5.21}$$

where and $k_1(x,t)$ and $k_2(x,t)$ are known functions which are called kernels of the integral equation and $f(x)$ is also a known function, while the unknown function $u(x)$ represents the approximate solution of the integral equation. Basic principle is that for conversion of the integral equation into equivalent differential equation with initial conditions. The conversion is achieved by the well-known Leibnitz rule (Wazwaz, 2011).

Numerical computational Procedure is as follows,

Step 1: Differentiating (5.21) twice with respect to x,using Leibnitz rule,we get differential equations with subject to initial conditions $u(0) = \beta, u'(0) = \gamma$.

Step 2: Applying Haar wavelet collocation method, Let us assume that,

$$u''(0) = \sum_{i=1}^{N} a_i\, h_i(x) \tag{5.22}$$

Step 3: By integrating (5.22) twice and substituting the initial conditions, we get,

$$u'(x) = \gamma + \sum_{i=1}^{N} a_i\, p_{1,i}(x) \tag{5.23}$$

$$u(x) = \beta + \gamma x + \sum_{i=1}^{N} a_i\, p_{2,i}(x) \tag{5.24}$$

Step 4: Substituting (5.22)-(5.24) in the differential equation, which reduces to the nonlinear system of N equations with N unknowns and then the Newton's method is used to obtain the Haar coefficientsa$_i$,$\mathrm{i} = 1,2,\cdots,\mathrm{N}$. Substituting Haar coefficients in (5.24) to obtain the required approximate solution of equation (5.21).

5.3. Numerical Examples

In this section, we consider some of the examples to demonstrate the capability of the presented method and error function is presented to verify the accuracy and efficiency of the following numerical results.

Example 5.1. Consider the following non-linear Volterra-Fredholm-Hammerstein integral equation given by.

$$\mathrm{u(x) = f(x) + \int_0^x (x-t)\,u^2(t)dt + \int_0^1 (x+t)\,u(t)dt}\,, \quad x\epsilon[0,1] \tag{5.25}$$

where $f(x) = \frac{-1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4}$, $k_1(x,t) = x - t$, $k_2(x,t) = x + t$ and the exact solution

$$u_e(x) = x^2 - 2 \qquad (5.26).$$

Step 1. Applying Leibnitz rule to convert IE to DE

$u' = f' + \frac{d}{dx}f_1(x) + \frac{d}{dx}f_2(x)$, we dropped λ_1 and λ_2 as they are unity

$$\frac{d}{dx}f_1(x) = k_1(x,t)u^2(x) + \int_0^x \frac{\partial k_1}{\partial x}u^2(x)dt$$

$$k_1(x,x) = 0,\ \frac{\partial k_1}{\partial x}=1$$

$$\frac{d}{dx}f_1(x) = \int_0^x u^2(x)\,dt$$

$$\frac{d}{dx}f_2(x) = \int_0^1 \frac{\partial k_2}{\partial x}u(t)\,dt\,;\ \frac{\partial k_2}{\partial x} = 1$$

$$\frac{d}{dx}f_2(x) = \int_0^1 u(t)\,dt$$

$$u'(x) = f'(x) + \int_0^x u^2(t)\,dt + \int_0^1 u(t)\,dt \qquad (5.27)$$

Now $u'' = f'' + \frac{d}{dx}\int_0^x u^2(t)\,dt + \frac{d}{dx}\int_0^1 u(t)\,dt$

$$\frac{d}{dx}G_1(x) = u^2(x) + \int_0^x \frac{\partial u(t)}{\partial x}dt = u^2(x)$$

$$\frac{d}{dx}G_1(x) = 0$$

$$U'' = f'' + u^2(x)$$

$$f' = \frac{1}{5}x^5 + \frac{4}{3}x^3 - 2x + \frac{5}{3}$$

$$f'' = -x^4 + 4x^2 - 2 \qquad (5.28)$$

$$U'' = u^2(x) - x^4 + 4x^2 - 2$$

But not IVP yet until we specify u_0 and u'_0. We use (5.25) and (5.27)

$$\mathrm{u_0} = \mathrm{u}(0) = \mathrm{f}(0) + 0 + \int_0^1 k_1(0,t)\,\mathrm{u(t)dt} = \frac{-5}{4} + \int_0^1 t(t^2-2)dt$$

$$= \frac{-5}{4} + \int_0^1 \mathrm{t}^3 dt - 2\int_0^1 tdt = \frac{-5}{4} + \frac{\mathrm{t}^4}{4}\Big[_0^1 + \frac{2}{2}\mathrm{t}^2\Big[_0^1$$

$$= \frac{-5}{4} + \frac{1}{4} - 1 = \frac{-5+1}{4} - 1 = -2$$

$$\mathrm{u_0} = -2$$

$$\mathrm{u'_0} = \mathrm{u'_0}(0) = \mathrm{f'}(0) + 0 + \int_0^1 u(t)dt$$

$$= \frac{5}{3} + \int_0^1 (t^2-2)dt = \frac{5}{3} + \int_0^1 t^2 dt - 2\int_0^1 dt$$

$$\frac{5}{3} + \frac{t^3}{3}\Big[_0^1 - 2 = \frac{5}{3} + \frac{1}{3} - 2 = 0$$

$$\mathrm{u'_0} = 0$$

Thus (5.25) now turns in to IVP given by

$$\mathrm{u''(x)} = \mathrm{u^2(\,t)} - x^4 + 4x^2 - 2;\ \mathrm{u_0} = -2,\ \mathrm{u'_0} = 0 \qquad (5.29)$$

Step 2. Expand $\mathrm{u''}$ in terms of finite number of Haar wavelets

$$\mathrm{U''(x)} = \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i\,h_i(x)} \qquad (5.30)$$

$\mathrm{N} = 2^{\mathrm{J}+1}$, J ≡maximum level of resolution of Haar wavelets.

Haar wavelets family

$$h_1(\mathrm{x}) = \begin{cases} 1, & \text{for } \mathrm{x} \in [0,1) \\ 0, & \text{elsewhere} \end{cases}$$

$$h_i(\mathrm{x}) = \begin{cases} 1, & for\ x \in [\alpha,\beta) \\ -1, & for\ x \in [\alpha,\beta), \\ 0, & elsewhere \end{cases} \qquad i = 2,3,4,\cdots$$

where $\alpha = \frac{\mathrm{k}}{\mathrm{m}}$; $\beta = \frac{\mathrm{k+1}}{\mathrm{N}}$; $\gamma = \frac{\mathrm{k+1}}{\mathrm{m}}$ with $\mathrm{m} = 2^{\mathrm{l}}$, $\mathrm{l} = 0,1,2,3,\cdots,\mathrm{J}$ and $k = 0,1,2,\cdots,\mathrm{m}-1$

index $''i''$ $\quad i_{min} = m_{min} + k_{min} + 1 = 1 + 0 + 1 = 2$

$$i_{max} = m_{max} + k_{max} + 1 = 2^{\mathrm{J}} + 2^{\mathrm{J}} - 1 + 1 = 2^{\mathrm{J}+1} = \mathrm{N}$$

We now integrate (6) once

$$\int du'(x) = \sum \mathrm{a_i} \int_0^{\mathrm{x}} \mathrm{h_i}\,(x')dx'$$

$$u'(x) - u'_0 = \textstyle\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{1,i}(x)} \qquad (5.31)$$

We now integrate (6) twice or (7) once

$$\mathrm{u(x) - u_0} = \textstyle\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i} \int_0^{\mathrm{x}} \mathrm{p_{1,i}(t)\,dt} = \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{2,i}(x)}$$

$$\mathrm{u(x)} + 2 = \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{2,i}(x)}$$

or

$$\mathrm{u(x)} = \textstyle\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{2,i}(x)} - 2 \qquad (5.32)$$

Now we have an approximate solution in terms of second order integrals of Haar wavelets with N unknown coefficients to be addressed in the next step.

Step 3. In determining $\{\mathrm{a_i}\}_{\mathrm{i=1}}^{\mathrm{N}}$ we plus (5.30) and (5.32) into IVP in (5.29) we obtain

$$\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i h_i(x)} = \left[\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{2,i}(x)} - 2\right]^2 - x^4 + 4x^2 - 2$$

or

$$\textstyle\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i h_i(x)} - \left[\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{2,i}(x)} - 2\right]^2 + x^2(x^2 - 4) + 2 = 0 \qquad (5.33)$$

It's clear that for fixed x (5.33) is a single equation of N unknown variables.

Step 4. Convert (5.33) into system of N nonlinear equations of $\{\mathrm{a_i}\}_{\mathrm{i=1}}^{\mathrm{N}}$ using collocation pont.

$$X_{\mathrm{j}} = \frac{\mathrm{j}-1/2}{\mathrm{N}},\ \mathrm{j} = 1,2,\cdots,\mathrm{N}$$

If we evaluate (5.33) at N collocation point x_{j} .we get system nonlinear equations

$$f_j(\mathrm{a_1},\cdots,\mathrm{a_N}) = \textstyle\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i h_i}(x_{\mathrm{j}}) - \left[\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i p_{2,i}}(x_{\mathrm{j}}) - 2\right]^2 + x_{\mathrm{j}}^2({x_{\mathrm{j}}}^2 - 4) + 2 \qquad (5.34)$$

It clears that $\mathrm{h_i}(x_\mathrm{j}) = [\mathrm{H_i}(\mathrm{N},\mathrm{N})]_\mathrm{j}$ element of Haar wavelet coefficient matrix.

and $\mathrm{p}_{2,\mathrm{i}}(x_\mathrm{j}) = [\mathrm{p}_2(\mathrm{N},\mathrm{N})]_{\mathrm{ij}}$ is element of operational matrix of second order.

Step 5. Algebraic system of non-linear equations into matrix equations

$$\bar{F}(\bar{a}) = \begin{pmatrix} \mathrm{f_1(a_1,a_2,\cdots,a_N)} \\ \mathrm{f_2(a_1,a_2,\cdots,a_N)} \\ \vdots \\ \mathrm{f_N(a_1,a_2,\cdots,a_N)} \end{pmatrix} = \bar{0} \qquad (5.35)$$

This nothing but a problem of root finding problem for multivariable function .In the next step we deal with numerical analysis of root finding for system of nonlinear equations with $\mathrm{a_1,a_2,\cdots,a_N}$ variables.

Step 6.Vectorial generalization of Newton methods root finding

Scalar $\mathrm{a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}}$ or $f'(a_0)\Delta a = -f(a_0)$

Vector $\overline{\mathrm{a_1}} = \overline{\mathrm{a_0}} + [\mathrm{DF}(\overline{\mathrm{a_0}})]^{-1}\bar{\mathrm{F}}(\overline{\mathrm{a_0}})$ or $(D\overline{\mathrm{F}})^{-1}\Delta\bar{\mathrm{a}} = -\bar{\mathrm{F}}(\overline{\mathrm{a_0}})$

where $\mathrm{D}\bar{\mathrm{F}}(\bar{\mathrm{a}})$ is the Jacobin of $\bar{\mathrm{F}}(\bar{\mathrm{a}})$ defined as

$$\mathrm{J(a)} = D\bar{\mathrm{F}}(\bar{\mathrm{a}}) = \begin{bmatrix} \frac{\partial \mathrm{f_1}}{\partial \mathrm{a_1}} & \cdots & \frac{\partial \mathrm{f_1}}{\partial \mathrm{a_N}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{f_N}}{\partial \mathrm{a_1}} & \cdots & \frac{\partial \mathrm{f_N}}{\partial \mathrm{a_N}} \end{bmatrix}_{\mathrm{M\times N}} \qquad (5.36)$$

The root finding iteration can be carried out in two different ways

$$\bar{\mathrm{a}}_{n+1} = \bar{\mathrm{a}}_n - \mathrm{J}^{-1}(\bar{\mathrm{a}}_n)\bar{\mathrm{F}}(\bar{\mathrm{a}}_n) \text{ or } \mathrm{J}(\bar{\mathrm{a}}_n)\Delta\bar{\mathrm{a}} = -\bar{\mathrm{F}}(\bar{\mathrm{a}}_n) \qquad 5.37)$$

The first one requires to compute J and its inverse which is costly in time and effort. But the second one is straight forward linear matrix equation that solves for $\Delta\bar{\mathrm{a}}$ such that

$$\bar{\mathrm{a}}_{n+1} = \bar{\mathrm{a}}_n + \Delta\bar{\mathrm{a}} \qquad (5.38)$$

We can easily solve for $\Delta\bar{\mathrm{a}}$ of

$$\mathrm{J}(\bar{\mathrm{a}}_n)\Delta\bar{\mathrm{a}} = -\bar{\mathrm{F}}(\bar{\mathrm{a}}_n)$$

using Gaussian elimination (improved one).

Computing of $J(\bar{a}_n)$ symbolically is possible but cumbersome and time consuming. So automatics are possible via numerical approach. In so doing finite difference methods of demotes can be used.

Forward difference $\frac{\partial \overline{F(a)}}{\partial a_i} \cong \frac{\bar{F}(a_1,\cdots,a_i+h_i,\cdots,a_N)-\bar{F}(a_1,a_2,\cdots,a_N)}{h}$

Backward difference $\frac{\partial \overline{F(a)}}{\partial a_i} \cong \frac{\bar{F}(a_1,\cdots,a_N)-\bar{F}(a_1,\cdots,a_i-h_i,\cdots,a_N)}{h}$

Central difference $\frac{\partial \overline{F(a)}}{\partial a_i} \cong \frac{\bar{F}(a_1,\cdots,a_i+h_i,\cdots,a_N)-\bar{F}(a_1,\cdots,a_i-h_i,\cdots,a_N)}{2h}$

But for this work we only use forward difference to complete Jacobian matrix.

Step 7. Numerical computational of $J(\bar{a})$ column-wise such that

$$J(\bar{a}) = [J(:,1) \mid J(:,2) \mid \cdots \mid J(:,N)]$$

where $J(:,i) = \left(\frac{\partial f_1}{\partial a_i}, \frac{\partial f_2}{\partial a_i}, \cdots \frac{\partial f_N}{\partial a_N}\right)^T = \frac{\partial \bar{F}\overline{(a)}}{\partial a_i}$ or $J(:,i) = \frac{\partial \bar{F}\overline{(a)}}{\partial a_i}$

and $\frac{\partial}{\partial a_i}\bar{F}\overline{(a)} \simeq \frac{\bar{F}(a_1,\cdots,a_i+h_i,\cdots,a_N)-\bar{F}(a_1,a_2,\cdots,a_N)}{h}$

where $h = \varepsilon$ is a very small but finite real constant.

Step 8. GEPP to solve for $\Delta\bar{a}(\bar{a}_{n+1})$ given $\bar{a}_n$.

Forward elimination with pp $N - 1$ steps of i^{th} columns.

Backward substitution: to solve for components of $\Delta\bar{a}$ starts from Δa_N through Δa_1

$$\Delta\bar{a} = (\Delta a_1, \Delta a_2, \cdots, \Delta a_N)^T.$$

Step 9. Stopping criteria for Newton method's iteration.

Two ways

1. $\|\bar{F}(\bar{a}_{n+1})\| = [\sum f_i^2]^2 \leq \varepsilon$ ε being tolerance
2. $\|\Delta a_1\| = \|\bar{a}_{n+1} - \bar{a}_n\| = [\sum \Delta a_i^2]^2 \leq \varepsilon$

Step 10. If step 9 is met with success, we evaluate $\bar{a}_{n+1}$ the improved root reach's

$$\bar{a}_{n+1} = \bar{a}_n + \Delta a$$

Step 11. Evaluate $U_a(x) \equiv U_a(x, \{a_1, a_2, \cdots, a_N\})$

$$U_a(x) = \sum a_i p_{2,i}(x) - 2$$

Step 12. Store $\{a_i\}$, U_e, U_a, $\| U_e - U_a \|$,plot $U_e$vs x, $U_a$vs x tabulates results for analysis and $u(x)$.

Therefore, the numerical result for example 1 using MATLAB code A.

Table 1: Numerical Results of Example 1 for $N = 4$

index	xj	Presented method(Error)	Uexact(xj)	Uapprox(xj)	Error
1	1.2500e-01	0.19866	-1.98e+00	-1.98e+00	1.78e-15
2	3.7500e-01	0.38933	-1.86e+00	-1.86e+00	5.25e-12
3	6.2500e-01	0.56400	-1.61e+00	-1.61e+00	3.66e-10
4	8.7500e-01	0.71466	-1.23e+00	-1.23e+00	4.65e-09

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would be decreases. The absolute errors for different values of N are shown in table 1. It is clear that the HWCM is more accurate for small values of N and, it seems that our method compared very well with those obtained via the presented method. For more clarity we can see the plot of these approximate values as follow.

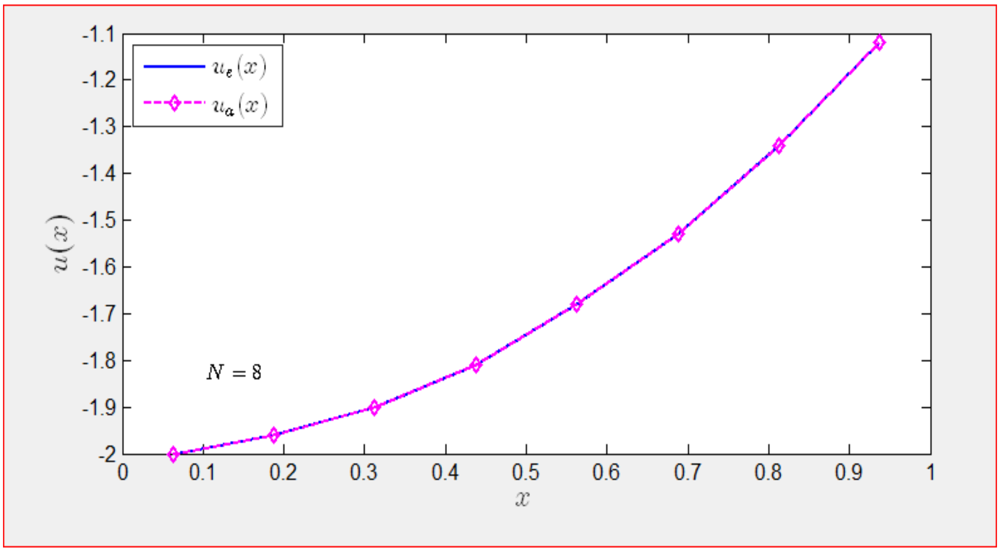

Figure 1. Numerical solutions of Example 1 of table 1 for $N = 8$.

Table 2: Numerical Results of Example 1 for $N = 16$

index	xj	a_i	Uexact(xj)	Uapprox(xj)	Error
1	3.1250e-02	2.00e+00	-2.00e+00	-2.00e+00	0.00e+00
3	1.5625e-01	2.54e-16	-1.98e+00	-1.98e+00	0.00e+00
5	2.8125e-01	-5.82e-16	-1.92e+00	-1.92e+00	0.00e+00
7	4.0625e-	-5.71e-15	-1.83e+00	-1.83e+00	0.00e+00
9	5.3125e-01	4.76e-16	-1.72e+00	-1.72e+00	0.00e+00
11	6.5625e-01	-5.96e-17	-1.57e+00	-1.57e+00	0.00e+00
13	7.8125e-01	-1.58e-15	-1.39e+00	-1.39e+0	2.22e-16
15	9.0625e-01	-1.12e-14-	-1.18e+00	-1.18e+00	8.88e-16

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would increases and computational time is increases.

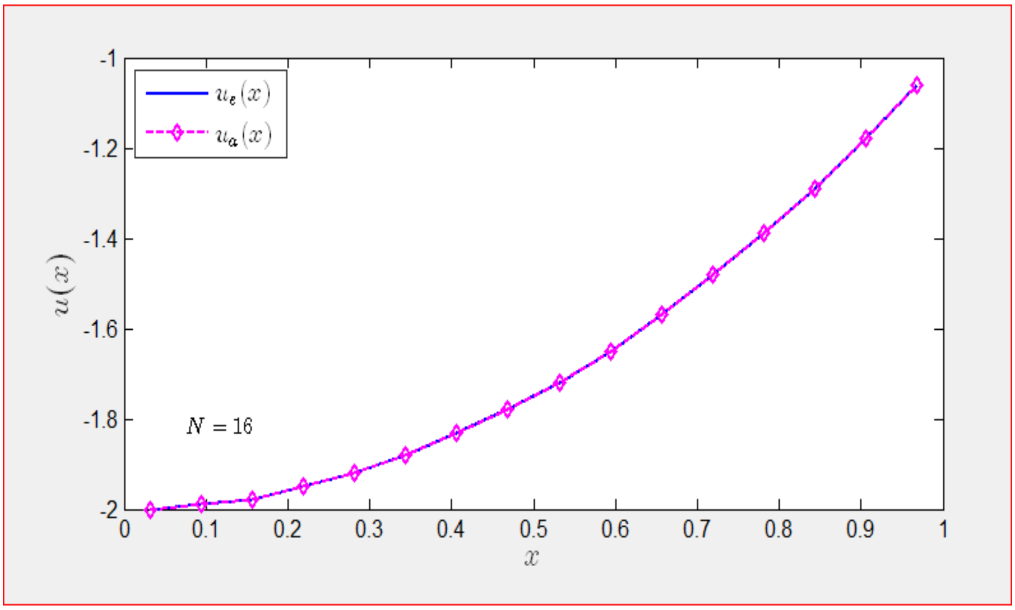

Figure 2. Numerical solutions of Example 1 of table 3 for N = 16.

Table 3. Error analysis of example 1.

N	Maximum Error	Computation Time(sec)
4	2.1765e-17	0.151731
8	8.3816e-31	0.271271
16	4.9304e-30	2.978046
32	1.6172e-29	50.831851
64	4.7369e-29	1000.663703

From Table 3 we could conclude that for different value of N the computational time increases and the error maximum (Emax) decreases.

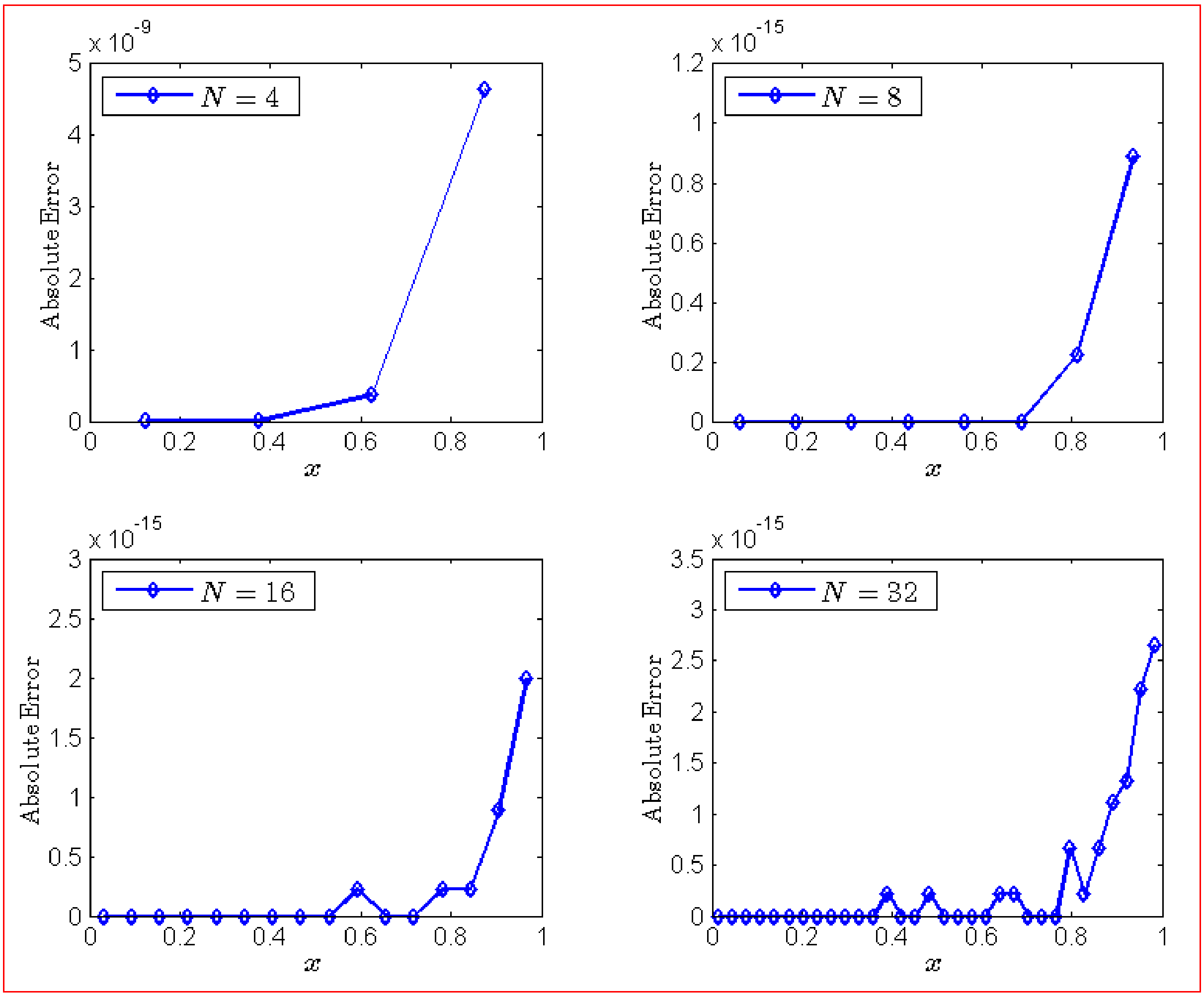

Figure 3**:** Absolute error of Example 1 of tables for variant N =4, N=8, N=16 and N=32. If we took more collocation points and we would get more accurate results. The collocation points are increases in each step lengths.

Examples 5.2. Consider the non-linear Volterra-Fredholm-Hammerstein integral equation is given by:

$$y(x) = 2\cos(x) - 2 + 3\int_0^x \sin(x-t)\, y^2(t)dt + \frac{6}{7-6\cos 1}\int_0^1 (1-t)\cos^2(x)\big(t + y(t)\big)dt$$

the exact solution is $y(x) = \cos(x)$.

Let $T_0 = \lambda_2 \cos^2(x)\int_0^1 (1-t)\,\big(t + y(t)\big)dt$, $\lambda_1 = 3$, $\lambda_2 = \frac{6}{7-6\cos 1}$,

$$T_1 = \int_0^1 \{t + u(t) - t^2 - ty(t)\}dt = \int_0^1 tdt - \int_0^1 t^2dt + \int_0^1 (1-t)y(t)dt$$

$$= \frac{t^2}{2}\Big[_0^1 - \frac{t^3}{3}\Big[_0^1 + \int_0^1 (1-t)y(t)dt = \frac{1}{6} + \int_0^1 (1-t)y(t)dt$$

$$T_1 = \frac{1}{6} + \int_0^1 (1-t)y(t)dt \equiv \text{Constant}$$

Since $y_e = \cos(t)$, $T_1 = \frac{1}{6} + \int_0^1 (1-t)\cos(t)dt = \frac{1}{6} + T_2$,because $T_2 = \int_0^1 (1-t)\cos(t)dt$

$$T_2 = \int_0^1 (1-t)\cos(t)dt = \int_0^1 \cos(t)dt - \int_0^1 t\cos(t)dt \text{ ,let } T_3 = \int_0^1 t\cos(t)dt$$

$$= \sin t \Big[_0^1 - T_3$$

Integration by Parts: let $v = t$, $du(sint) = \cos(t)\,dt = d(sint)$, $u = \sin t$

$$T_3 = \int v du = uv\Big[_0^1 - \int_0^1 u dv = t\sin(t)\Big[_0^1 - \int_0^1 \sin(t)\,dt$$

$$= t\sin(t)\Big[_0^1 + \cos t\Big[_0^1$$

$$T_3 = \sin 1 + \cos 1 - 1 \text{ is constant}$$

$$T_2 = \sin t\Big[_0^1 - \sin 1 - \cos 1 + 1 = \sin 1 - \sin 1 - \cos 1 + 1 = 1 - \cos 1$$

$$T_2 = 1 - \cos 1$$

$$T_1 = \frac{1}{6} + 1 - \cos 1 = \frac{7 - 6\cos 1}{6}$$

$$T_1 = \frac{7 - 6\cos 1}{6}$$

$$T_0 = \lambda_2 \cos^2(x) T_1, \lambda_2 = \frac{6}{7 - 6\cos 1}, \text{ so } \lambda_2 T_1 = 1$$

$$T_0 = \cos^2(x)$$

Now $y(x) = 2\cos(x) - 2 + \lambda_2 T_1 \cos^2(x) + \lambda_1 \int_0^x \sin(x-t) y^2(t)\,dt$

$y(x) = g(x) + \lambda_1 \int_0^x \sin(x-t) y^2(t)dt$, $g(x) = 2cosx - 2 + \lambda_2 T_1 \cos^2(x)$

Now $y'(x) = g'(x) + \lambda_1 \frac{d}{dx}\int_0^x \sin(x-t)\, y^2(t)dt$

$$= g'(x) + \lambda_1 \int_0^x \cos(x-t)\, y^2(t)dt$$

$$y'(x) = g'(x) + \lambda_1 \int_0^x \cos(x-t)\, y^2(t)dt$$

Now $y''(x) = g''(x) + \lambda_1 \frac{d}{dx}\int_0^x \cos(x-t)\, y^2(t)dt$

$$\frac{d}{dx} H = \cos(x-t)\, y^2(t) - \int_0^x \sin(x-t)\, y^2(t)\,dt$$

$$= y^2(t) - \int_0^x \sin(x-t)\, y^2(t)\,dt$$

$$y''(x) = g''(x) + \lambda_1 \int_0^x \sin(x-t) y^2(t)dt + \lambda_1 y^2 \tag{5.39}$$

But

$$y(x) = g(x) + \lambda_1 \int_0^x \sin(x-t)\, y^2(t)dt \tag{5.40}$$

From (5.39) and (5.40)

$$y''(x) + y(x) = g'' + g + \lambda_1 y^2$$

or $y''(x) + y(x) - g''(x) - g(x) - \lambda_1 y^2, y_0, y'_0$

$$g(x) = 2\cos x - 2 + \lambda_2 T_1 \cos^2 x$$

$$g'(x) = -2\sin x - \lambda_2 T_1 \sin 2x$$

$$g''(x) = -2\cos x - 2\lambda_2 T_1 \cos 2x$$

$$g'''(x) = 2\cos x + 4\lambda_2 T_1 \sin 2x$$

Better yet $y''(x) + y(x) - \lambda_1 y^2(x) - g''(x) - g(x) = 0, \lambda_1 = 3$

$$y_0 = g''(0) = 2\cos(0) - 2 + \lambda_2 T_1 \cos^2(0) = \lambda_2 T_1 = 1$$

$$y_0 = 1$$

$$T_1 = \lambda_2^{-1}, \cos(0) = 1$$

$y'_0 = g'(0) = -2\sin(0) - \lambda_2 T_1 \sin 0 = 0$, since sin (0) =0

$$y'_0 = 0.$$

Now $y''(x) + y(x) - \lambda_1 y^2(x) - g''(x) - g(x) = 0, y_0 = 1, y'_0 = 0$ is IVP

Expand y'' in terms of Haar wavelets.

$$y''(x) = \sum_{i=1}^{N} a_i h_i(x), \quad y' - y'_0 = \sum a_i p_{1,i}(x),$$

$y' = \sum a_i p_{1,i}(x)$ because $y'_0 = 0$

and $y(x) - y_0 = \sum a_i p_{2,i}(x)$, $y_a(x) = 1 + \sum a_i p_{2,i}(x)$

Approximate solution

$$y_a(x) = 1 + \sum a_i p_{2,i}(x) \text{ , } \{a_i\}_{i=1}^{N}$$

Now $y' + y - g'' - g = \lambda_1 y^2(x) = 3y^2(x)$

$$\sum_{i=1}^{N} a_i h_i(x) + \left\{1 + \sum a_i p_{2,i}(x)\right\} - g''(x) - g(x)$$

$$= 3\left[1 + \sum a_i p_{2,i}(x)\right]^2$$

$$g''(x) + g(x) = -2\cos x - 2\lambda_2 T_1 \cos 2x + 2\cos x - 2 + \lambda_2 T_1 \cos^2 x$$

$= -2 - 2\lambda_2 T_1 \cos 2x + \lambda_2 T_1 \cos^2 x$, since $\lambda_2 T_1 = 1$

$$= -2 - \lambda_2 T_1 \{2\cos 2x - \cos^2 x\}, \ \lambda_2 T_1 = 1$$

$$= -2 - \{2\cos^2 x - 2\sin^2 x - \cos^2 x\}$$

$$= -2 - \{1 - \sin^2 x - 2\sin^2 x\}$$

$$= -2 - \{1 - 3\sin^2 x\}$$

$$= -2 - 1 + 3\sin^2 x = -3 + 3\sin^2 x$$

$$= -3(1-\sin^2 x) = -3\cos^2(x)$$

Therefore $g''(x) + g(x) = -3\cos^2(x)$

$$\sum_{i=1}^{N} a_i\, h_i(x) + \left[1 + \sum a_i p_{2,i}(x)\right] - 3\left[1 + \sum a_i p_{2,i}(x)\right]^2 + 3\cos^2(x) = 0$$

We now introduce collocation points $x_j = \frac{(j-1/2)}{N}$ to produce system of N nonlinear equations to solve for $\{a_1, a_2, \cdots, a_N\}$. Let's denote

$f_j(a_1, a_2, \cdots, a_N) = \sum_{i=1}^{N} a_i\, h_i(x) + \left[1 + \sum a_i p_{2,i}(x)\right] - 3\left[1 + \sum a_i p_{2,i}(x)\right]^2 + 3\cos^2(x)$ $, j = 1, 2, \cdots, N$ such that $\{f_j(a_1, a_2, \cdots, a_N)\}_{j=1}^{N}$ N is number of nonlinear equations.

Matrix equations

$$\bar{F}(\bar{a}) = \begin{pmatrix} f_1(a_1, a_2, \cdots, a_N) \\ f_2(a_1, a_2, \cdots, a_N) \\ \vdots \\ f_N(a_1, a_2, \cdots, a_N) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ or $\bar{F}(\bar{a}) = \bar{0}$ equation need to be solved for roots.

Now for method

$$\bar{a}_{n+1} = \bar{a}_n - J^{-1}(\bar{a}_n)\bar{F}(\bar{a})$$

Even better, $J(\bar{a})\Delta\bar{a} = -\bar{F}(\bar{a})$ linear equation with $\{\Delta a_1, \Delta a_2, \cdots, \Delta a_N\}$ variables

Gaussian elimination with partial pivoting can handle this

$$J(\bar{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial a_1} & \cdots & \frac{\partial f_1}{\partial a_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial a_1} & \cdots & \frac{\partial f_N}{\partial a_N} \end{bmatrix} = [J(:,1), J(:,2), \cdots, J(:,N)]$$

$J(:,i) = \frac{\partial}{\partial a_i}\bar{F}(\bar{a}) \simeq \frac{\bar{F}(a_1,\cdots,a_i+h_i,\cdots,a_N) - \bar{F}(a_1,a_2,\cdots,a_N)}{h}$ forward difference to compute J_{ij}'s

$$\bar{a}_{n+1} = \bar{a}_n + \Delta\bar{a}$$

$\|F(\bar{a}_{n+1})\| \le \varepsilon$ or $\|(\bar{a}_{n+1} - \bar{a}_n)\| \le \varepsilon$ stopping criteria for iteration

Plot and display $U_e(x)$, $U_a(x)$, $\|U_e(x) - U_a(x)\|$

$$y_e(x) = \cos(x),\ y_a(x) = \sum a_i p_{2,i}(x) + 1$$

$$f_j(a_1, a_2, \cdots, a_N) = \sum a_i\left\{h_i(x_j) + p_{2,i}(x_j)\right\} + 3\cos^2(x_j) - 3\left[1 + \sum a_i p_{2,i}(x_j)\right]^2$$

Again

$$f_j(a_1, a_2, \cdots, a_N) = \sum a_i h_i(x_j) + [1 + \sum a_i p_{2,i}(x_j)] - 3[1 + \sum a_i p_{2,i}(x_j)]^2 + 3\cos^2(x_j)$$

$$f_j(a_1, a_2, \cdots, a_N) = \sum a_i h_i(x_j) + \{1 + \sum a_i p_{2,i}(x)\} - 3[1 + \sum a_i p_{2,i}(x_j)]^2 + 3\cos^2(x_j)$$

Therefore, the numerical result for example 2 using MATLAB code B.

Table 4: Numerical Results of example 2 for $N = 8$

Index	Present Method	HWCM			
n	Error	xj	Uexact(xj)	Uapprox(xj)	Error
1	1.0572e-0 004	6.2500e-02	9.98e-01	9.98e-01	3.21e-06
2	7.7160e- 006	1.8750e-01	9.82e-01	9.82e-01	1.37e-05
3	3.8609e -005	3.1250e-01	9.52e-01	9.52e-01	3.53e-05
4	1.4363e- 005	4.3750e-01	9.06e-01	9.06e-01	6.92e-05
5	3.0035e -005	5.6250e-01	8.46e-01	8.46e-01	1.17e-04
6	4.1888e- 005	6.8750e-01	7.73e-01	7.73e-01	1.81e-04

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would increases. The absolute errors for different values of N are shown in table 1. It is clear that the HWCM is more accurate for small values of N and, and it seems that our method compared very well with those obtained via the presented method.

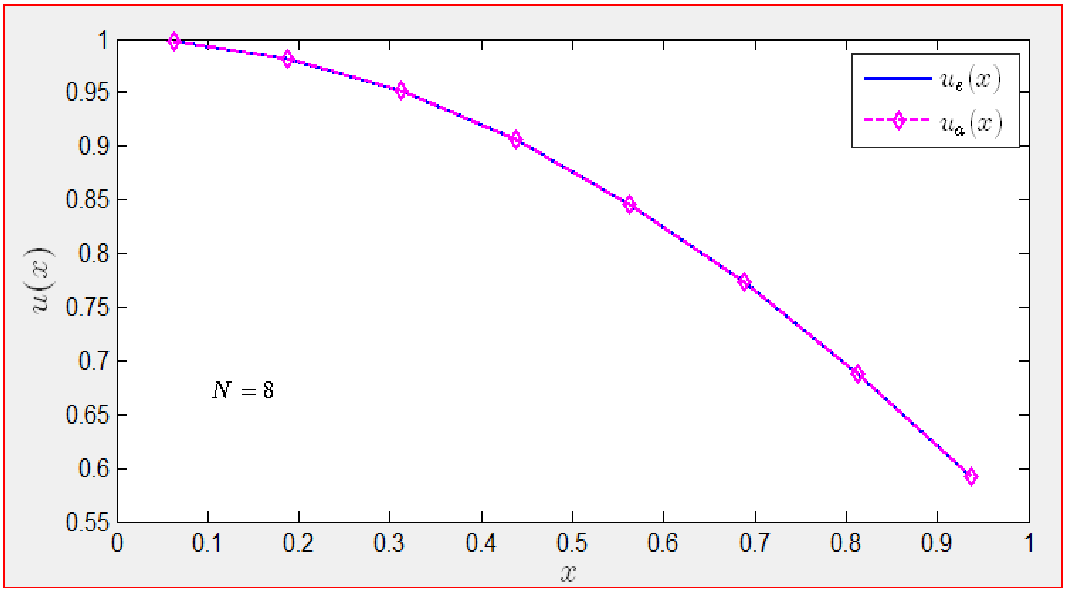

Figure 4. Numerical solutions of example 2 of table 7 with N = 8.

Table 5: Numerical Results of Example 2 for N = 16

index	xj	a_i	Uexact(xj)	Uapprox(xj)	Error
1	3.1250e-02	-8.42e-01	1.00e+00	1.00e+00	1.99e-07
3	1.5625e-01	-3.08e-02	9.88e-01	9.88e-01	2.13e-06
5	2.8125e-01	-7.79e-03	9.61e-01	9.61e-01	6.75e-06
7	4.0625e-01	-3.65e-02	9.19e-01	9.19e-01	1.43e-05
9	5.3125e-01	-1.95e-03	8.62e-01	8.62e-01	2.53e-05
11	6.5625e-01	-9.61e-03	7.92e-01	7.92e-01	4.00e-05
13	7.8125e-01	-1.67e-02	7.10e-01	7.10e-01	5.92e-05
15	9.0625e-01	-2.27e-02	6.17e-01	6.17e-01	8.31e-05

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would increases.

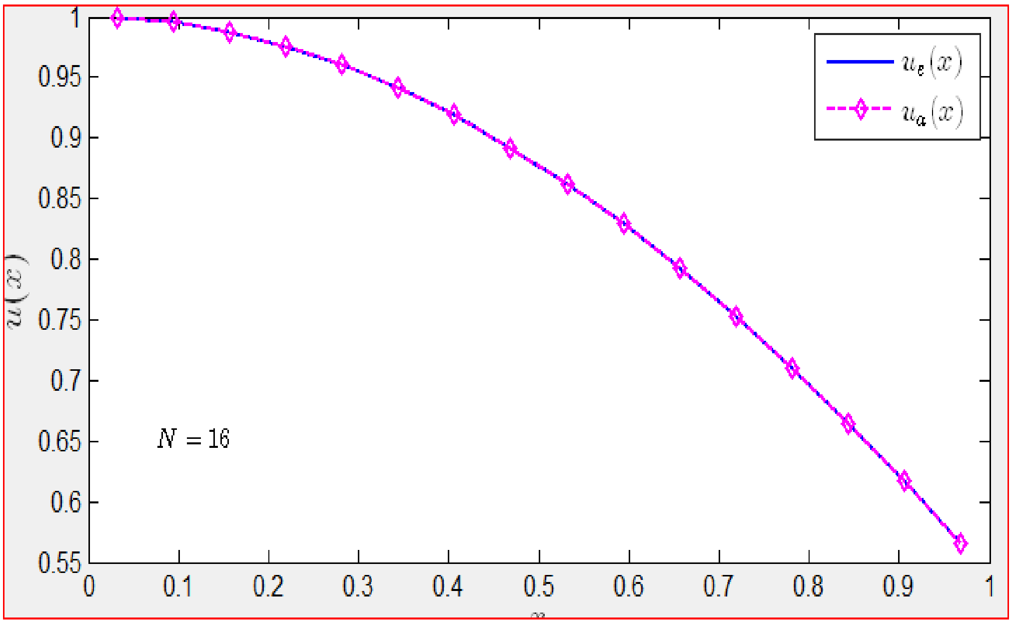

Figure 5. Numerical solutions of example 2 of table 8 with N = 16.

Table 6. Error analysis of example 2.

N	Present method (Maximum error)	Maximum Error	Computation Time (sec)
4	1.7367e-005	2.2291e-06	0.151731
8	1.9265e-012	2.5531e-07	0.271271
16	1.1718e-015	3.121e-08	2.978046
32	3.4760e-019	3.8794e-09	50.831851

From Table 6 we could conclude that for different value of N the computational time is increases and the error maximum is decreases. The computational maximum absolute errors for different values of N are shown in Table 6. It is clear that the proposed method is more accurate for small values of N. it seems that our method compared very well with those obtained via the presented method.

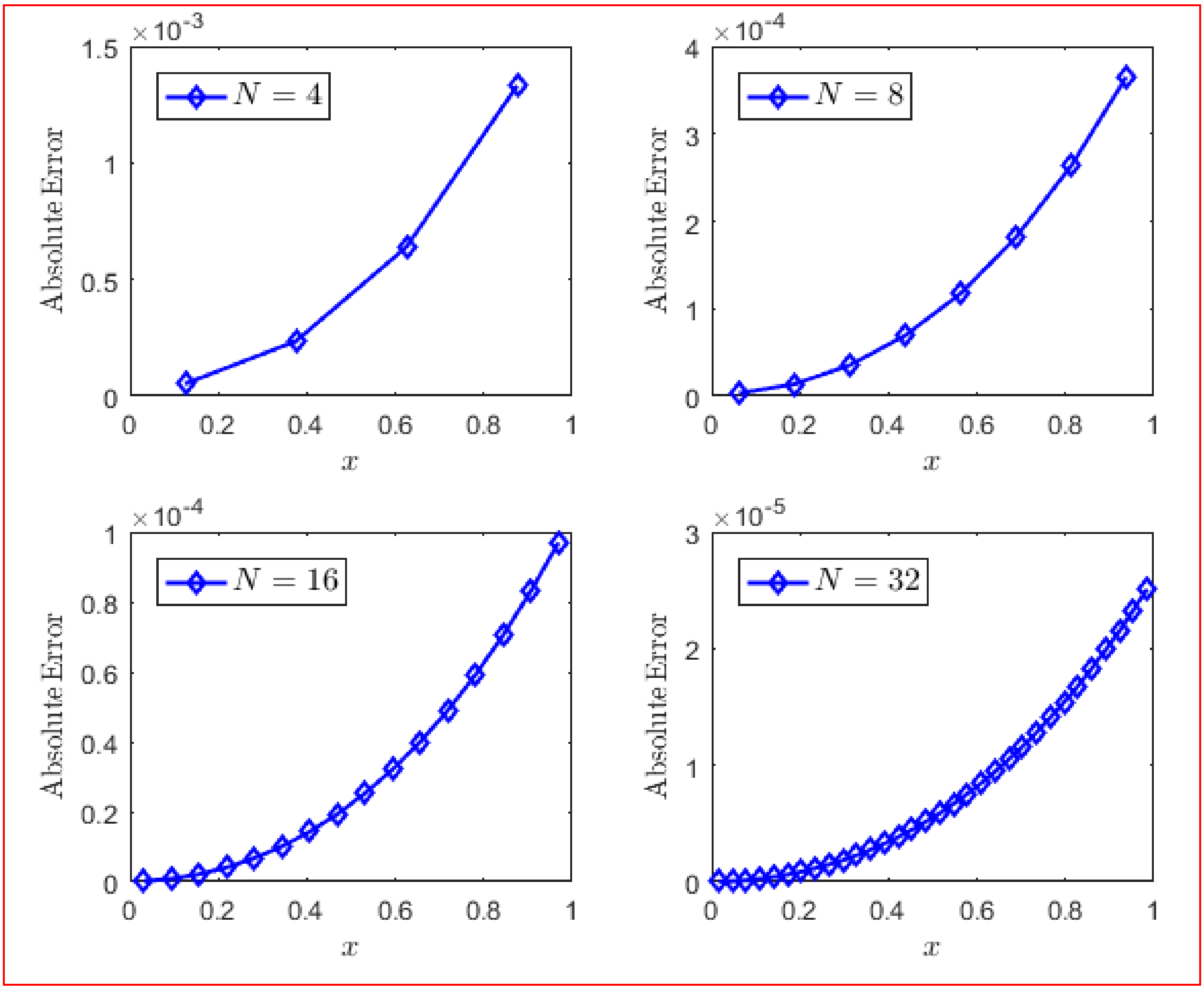

Figure 6: Absolute error of example 2 of above table for N =4, N=8, N=16 and N=32. If we took more collocation points and we would get more accurate results.

Example 5. Consider the following non-linear Volterra-Fredholm-Hammerstein integral equation given by.

$$u(x) = \frac{x}{2} - \frac{x^4}{12} - \frac{1}{3} + \int_0^1 (x-t)u(t)dt + \int_0^x (x+t)\,u^2(t)dt \qquad (5.41)$$

where $f(x) = \frac{x}{2} - \frac{x^4}{12} - \frac{1}{3}$, $k_1(x,t) = x - t$, $k_2(x,t) = x + t$ and exact solution $u(x) = x$.

Solution: Step 1. Apply Leibnitz rule to convert integral equation to differential equation.

Leibnitz rule:

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(x,t)dt = f\big(x, g(x)\big)\frac{dg}{dx} - f\big(x, h(x)\big)\frac{dh}{dx} + \int_{h(x)}^{g(x)} \frac{\partial f}{\partial x} dt \qquad (5.42)$$

Differentiate (1) once

$$u'(x) = \frac{du}{dx} = f'(x) + \frac{d}{dx}\int_0^x k_1(x,t)u^2(t)dt + \frac{d}{dx}\int_0^1 k_2(x,t)\,u(t)dt$$

where $f'(x) = \left(\frac{x}{2} - \frac{x^4}{12} - \frac{1}{3}\right)' = \frac{1}{2} - 4\frac{x^3}{12} = \frac{1}{2} - \frac{x^3}{3}$

Leibnitz rule also gives

$$\frac{d}{dx}\int_0^x k_1(x,t)u^2(t)dt = k_1(x,x)u^2(x) - k_1(x,0)u^2(0)0 + \int_0^x \frac{\partial k_1}{\partial x}u^2(x)dt$$

$$= 0 - 0 + \int_0^x u^2(x)dt = \int_0^x u^2(x)dt$$

and $\frac{d}{dx}\int_0^1 k_2(x,t)u(t)dt = k_1(x,1)u(1)0 - k_1(x,0)u(0)0 + \int_0^1 \frac{\partial k_2}{\partial x}u(x)dt$

$$= 0 - 0 + \int_0^1 u(t)dt = \int_0^1 u(t)dt$$

$$u'(x) = \frac{1}{2} - \frac{x^3}{3} + \int_0^x u^2(t)dt + \int_0^1 u(t)dt$$

Still the integral terms survive, so we need to differentiate (1) twice

$$u''(x) = -3\frac{x^2}{3} + \frac{d}{dx}\int_0^x u^2(x)dt + \frac{d}{dx}\int_0^1 u(x)dt \tag{5.43}$$

Leibnitz rule again

$$\frac{d}{dx}\int_0^x u^2(x)dt = u^2(x)\frac{dx}{dx} - u^2(0)\frac{d0}{dx} + \int_0^x \frac{\partial}{\partial x}u^2(0)dt$$

$$= u^2(x)$$

and

$$\frac{d}{dx}\int_0^1 u(x)dt = u(1)\frac{d1}{dx} - u(1)\frac{d1}{dx} + \int_0^1 \frac{\partial}{\partial x}u(1)dt = 0$$

$$u''(x) = -x^2 + u^2(x)$$

or

$$u''(x) - u^2(x) + x^2 = 0$$

By using (1) at $x = 0$

$$u(0) = -\frac{1}{3} + \int_0^1 (0 + t)u(t)\,dt + 0$$

$$= -\frac{1}{3} + \int_0^1 tu(t)\,dt = -\frac{1}{3} + \int_0^1 t^2\,dt$$

$$u(0) = -\frac{1}{3} + \left.\frac{t^3}{3}\right|_0^1 = -\frac{1}{3} + \frac{1}{3} = 0$$

$$\mathrm{u}(0) = 0$$

$$\mathrm{u}'(0) = \frac{1}{2} + \int_0^1 0\,dt + \int_0^1 t\,dt = \frac{1}{2} + 0 + \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\mathrm{u}'(0) = 1$$

We now have IVP

$$\mathrm{u}''(\mathrm{x}) - \mathrm{u}^2(\mathrm{x}) + x^2 = 0, \mathrm{u}(0) = 0, \mathrm{u}'(0) = 1 \tag{5.44}$$

Step 2: Collocation method using Haar wavelets

Let $$\mathrm{U}''(\mathrm{x}) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{h_i}(\mathrm{x}) \tag{5.45}$$

where Haar wavelets are defined as

$$h_1(\mathrm{x}) = \begin{cases} 1, & \text{for } \mathrm{x} \in [0,1) \\ 0, & \text{elsewhere} \end{cases}$$

$$h_i(\mathrm{x}) = \begin{cases} 1, & for\ x \in [\alpha, \beta) \\ -1, & for\ x \in [\alpha, \beta), \\ 0, & elsewhere \end{cases} \quad i = 2,3,4,\cdots$$

If we integrate (5), once we have

$$\frac{d}{dx}\mathrm{U}''(\mathrm{x}) = \sum \mathrm{a_i} \int_0^1 \mathrm{h_i}(\mathrm{x}')\mathrm{dx}'$$

$$u'(x) - u'(0) = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{a_i} \int_0^1 \mathrm{h_i}(\mathrm{x})\mathrm{dx}$$

But $\int_0^1 \mathrm{h_i}(\mathrm{x})\mathrm{dx} = \mathrm{p_{1,i}}(\mathrm{x})$

$$u'(x) = u'(0) + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{1,i}}(\mathrm{x})$$

or
$$u'(x) = 1 + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{1,i}}(\mathrm{x}) \qquad (5.46)$$

Integrating twice

$$u(x) = u(0) + \int_0^x [1 + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{1,i}}(\mathrm{x})]\,dx$$

$$u(x) = \int_0^x 1dt + \sum \mathrm{a_i} \int_0^{\mathrm{x}} \mathrm{p_{1,i}}(\mathrm{x})\,dt$$

or

$$u(x) = x + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{2,i}}(\mathrm{x}) \qquad (5.47)$$

Step3: We plug (5.45)-(5.49) into (5.46) gives

$$\sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{h_i}(\mathrm{x}) - \left[x + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{2,i}}(\mathrm{x})\right]^2 + x^2 = 0 \qquad (5.48)$$

This non-linear integral algebraic equations with N unknowns $\{\mathrm{a_i}\}_{i=1}^{N}$.

Step 4: Produce N of nonlinear equation using collocation points

$$\mathrm{x_j} = \frac{\mathrm{j} - \frac{1}{2}}{\mathrm{N}}$$

That is $\mathrm{p_{2,i}}(x_j)$ becomes an element of an $\mathrm{N \times N}$ operational matrix $\mathrm{p_2}$.

$$\sum \mathrm{a_i h_i}(x_j) - \left[x_j + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{2,i}}(x_j)\right]^2 + {x_j}^2 = 0, \mathrm{j} = 1,2,3,\ldots,\mathrm{N} \qquad (5.49)$$

Now we have N equations nonlinear in $\{\mathrm{a_i}\}_{i=1}^{N}$.

Let $f_j(\mathrm{a_1, a_2, \cdots, a_N}) = \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{h_i}(x_j) - \left[x_j + \sum_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{a_i}\,\mathrm{p_{2,i}}(x_j)\right]^2 + {x_j}^2$

Step 5: Matrix equation

and
$$\bar{F}(\bar{a}) = \begin{pmatrix} \mathrm{f_1(a_1, a_2, \cdots, a_N)} \\ \mathrm{f_2(a_1, a_2, \cdots, a_N)} \\ \vdots \\ \mathrm{f_N(a_1, a_2, \cdots, a_N)} \end{pmatrix} \qquad (5.50)$$

Thus, (9) can be written as a matrix equation

$$\bar{\mathrm{F}}(\bar{\mathrm{a}}) = \bar{0} \tag{5.51}$$

Step 6: System of nonlinear algebraic equations can be solved numerically using newtons methods of root finding.

$$\mathrm{a}_{\mathrm{n}+1} = \mathrm{a}_{\mathrm{n}} - \frac{\mathrm{f}(\mathrm{a}_{\mathrm{n}})}{\mathrm{f}'(\mathrm{a}_{\mathrm{n}})}$$

$$\mathrm{a}_{\mathrm{n}+1} - \mathrm{a}_{\mathrm{n}} = -\mathrm{f}(\mathrm{a}_{\mathrm{n}})[\mathrm{f}'(\mathrm{a}_{\mathrm{n}})]^{-1}$$

$$\Delta\mathrm{a}_{\mathrm{n}} = -\mathrm{f}(\mathrm{a}_{\mathrm{n}})[\mathrm{f}'(\mathrm{a}_{\mathrm{n}})]^{-1}$$

or

$$[\mathrm{f}'(\mathrm{a}_{\mathrm{n}})]^{-1}\Delta\mathrm{a}_{\mathrm{n}} = -\mathrm{f}(\mathrm{a}_{\mathrm{n}})$$

For vector variables we make the following

$$\bar{\mathrm{a}}_{\mathrm{n}+1} = \bar{\mathrm{a}}_{\mathrm{n}} - [\mathrm{DF}]^{-1}\bar{\mathrm{F}}(\bar{\mathrm{a}})$$

Where $\mathrm{DF}(\bar{\mathrm{a}}_{\mathrm{n}}) = \mathrm{J}(\bar{\mathrm{a}}_{\mathrm{n}})$ is the Jacobian matrix of $\mathrm{N} \times \mathrm{N}$ dimension

$$\bar{\mathrm{a}}_{\mathrm{n}+1} = \bar{\mathrm{a}}_{\mathrm{n}} - \mathrm{J}^{-1}(\bar{\mathrm{a}})\bar{\mathrm{F}}(\bar{\mathrm{a}}) \tag{5.52}$$

This newton method to solve for N unknown coefficient $\{\mathrm{a}_{\mathrm{i}}\}_{i=1}^{N}$.

Finding inverse is coasty in Matlab, so we rewrite (5.52) by letting $\Delta\bar{\mathrm{a}} = \bar{\mathrm{a}}_{\mathrm{n}+1} - \bar{\mathrm{a}}_{\mathrm{n}}$

$$\Delta\bar{\mathrm{a}} = -\mathrm{J}^{-1}(\bar{\mathrm{a}})\bar{\mathrm{F}}(\bar{\mathrm{a}})$$

or

$$\mathrm{J}(\bar{\mathrm{a}})\Delta\bar{\mathrm{a}} = -\bar{\mathrm{F}}(\bar{\mathrm{a}}) \tag{5.53}$$

This is just a system of linear equations for which $\Delta\bar{\mathrm{a}}$ is the unknown vector once $\Delta\bar{\mathrm{a}}$ is solved, we can easily find $\bar{\mathrm{a}}_{\mathrm{n}+1}$ using the formula

$$\bar{\mathrm{a}}_{\mathrm{n}+1} = \bar{\mathrm{a}}_{\mathrm{n}} + \Delta\bar{\mathrm{a}} \tag{5.54}$$

Step 7: Numerical estimation of $\mathrm{J}(\bar{\mathrm{a}})$

$$\mathrm{J}(\bar{\mathrm{a}}) = \begin{bmatrix} \frac{\partial \mathrm{f}_1}{\partial \mathrm{a}_1} & \cdots & \frac{\partial \mathrm{f}_1}{\partial \mathrm{a}_{\mathrm{N}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathrm{f}_{\mathrm{N}}}{\partial \mathrm{a}_1} & \cdots & \frac{\partial \mathrm{f}_{\mathrm{N}}}{\partial \mathrm{a}_{\mathrm{N}}} \end{bmatrix}$$

$$\equiv [\mathrm{J}(:,1), \mathrm{J}(:,2), \cdots, \mathrm{J}(:,\mathrm{N})] \tag{5.55}$$

where
$$\mathrm{J}(:,i) = \begin{bmatrix} \frac{\partial \mathrm{f}_1}{\partial \mathrm{a_i}} \\ \frac{\partial \mathrm{f}_2}{\partial \mathrm{a_i}} \\ \vdots \\ \frac{\partial \mathrm{f_N}}{\partial \mathrm{a_i}} \end{bmatrix} = \frac{\partial}{\partial \mathrm{a_i}} \bar{\mathrm{F}}(\bar{\mathrm{a}}) = \mathrm{G_i}(\bar{\mathrm{a}}) \tag{5.56}$$

Finite difference method to find approximation values of partial derivatives.

$$\mathrm{G_i}(\bar{\mathrm{a}}) = \mathrm{J}(:,i) \cong \frac{\bar{\mathrm{F}}(\mathrm{a}_1,\mathrm{a}_2,\cdots,\mathrm{a_i}+\mathrm{h},\cdots,\mathrm{a_N})\bar{\mathrm{F}}(\bar{\mathrm{a}})}{\mathrm{h}} \tag{5.57}$$

Where h is a finite real constant. Notice only i^{th} component of $\bar{\mathrm{a}}$ is updated

Again
$$\mathrm{J}(:,i) \cong \frac{\bar{\mathrm{F}}(\mathrm{a}_1,\mathrm{a}_2,\cdots,\mathrm{a_i}+\mathrm{h},\cdots,\mathrm{a_N})\bar{\mathrm{F}}(\mathrm{a}_1,\mathrm{a}_2,\cdots,\mathrm{a_N})}{\mathrm{h}} \tag{5.58}$$

We complete Jacobian matrix by numerically solving one column at a time from j = 1, through N.

Step 8: Use GEPP to numerically solve for $\Delta\bar{\mathrm{a}}$ of (5.55)

1. Forward elimination with pp $(\mathrm{N}-1$ steps$)$ convert augmented matrix $[\mathrm{J}((\bar{\mathrm{a}})) \vdots -\bar{\mathrm{F}}(\bar{\mathrm{a}})]$ into UT matrix of $\mathrm{J}((\bar{\mathrm{a}}))$.

2. Back substitution to determine $\Delta\bar{\mathrm{a}}$ starting from $\bar{\mathrm{a}}_\mathrm{N}$, $\bar{\mathrm{a}}_{\mathrm{N}-1}$, $\bar{\mathrm{a}}_{\mathrm{N}-2}$, $\cdots$,$\bar{\mathrm{a}}_3$, $\bar{\mathrm{a}}_2$, $\bar{\mathrm{a}}_1$

Once $\Delta\bar{\mathrm{a}}$ is found calculate $\bar{\mathrm{a}}_{\mathrm{n}+1}$ (improved root)

$$\bar{\mathrm{a}}_{\mathrm{n}+1} = \bar{\mathrm{a}}_\mathrm{n} + \Delta\bar{\mathrm{a}}$$

Step 9: Check for convergence of the method using error analysis

$||\bar{\mathrm{F}}(\bar{\mathrm{a}}_{\mathrm{n}+1})|| \leq \varepsilon$, or $||\Delta\bar{\mathrm{a}}|| \leq \varepsilon$, ε: tolerance level

This is based on stopping criteria for Newton's method. Once an approximation value of $\bar{\mathrm{a}}$ is found, we estimate approximation value of u(x) using (7).

$$u(x) = x + \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{a_i}\, \mathrm{p_{2,i}}(\mathrm{x})$$

Therefore, the numerical result for example 3 using MATLAB code C.

Table 7: Numerical Results of example 3 for N = 8

index	xj	a_i	exact(xj)	Uapprox(xj)	AbsError
1	6.2500e-02	-1.17e-11	6.25e-02	6.25e-02	0.00e+00
2	1.8750e-01	1.17e-11	1.88e-01	1.88e-01	0.00e+00
3	3.1250e-01	-1.30e-18	3.13e-01	3.13e-01	0.00e+00
4	4.3750e-01	2.34e-11	4.38e-01	4.38e-01	0.00e+00
5	5.6250e-01	3.82e-20	5.63e-01	5.63e-01	0.00e+00
6	6.8750e-01	-2.39e-18	6.88e-01	6.88e-01	0.00e+00
7	8.1250e-01	1.12e-14	8.13e-01	8.12e-01	5.66e-15
8	9.3750e-01	4.41e-11	9.38e-01	9.37e-01	2.21e-13

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would decrease. Clearly to see this, we have a plot of these approximate solutions with the exact values as in Figure 7.

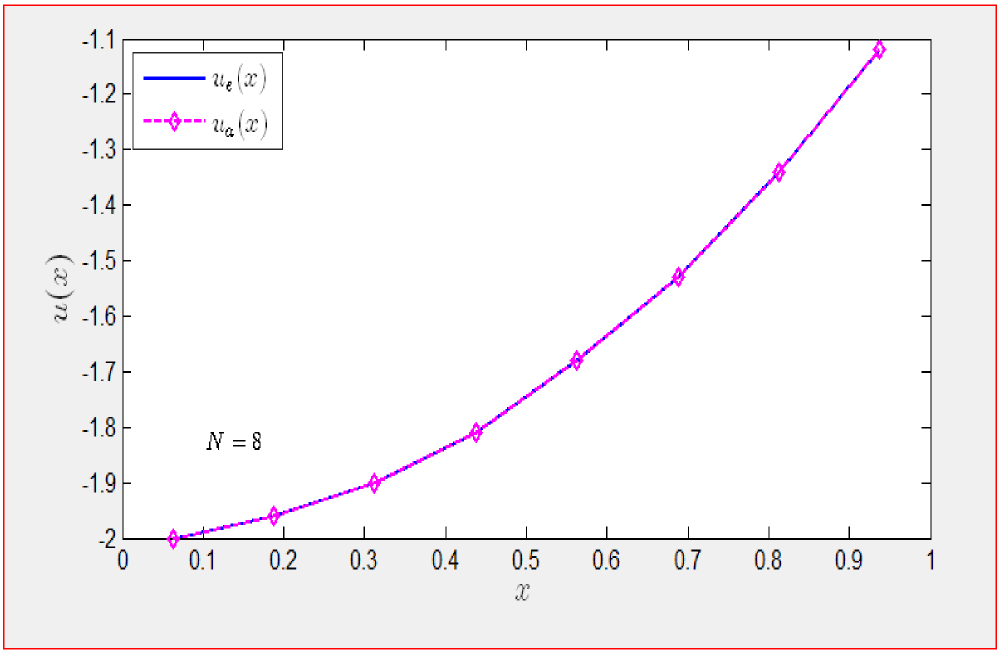

Figure 7. Numerical solutions of example 3 of table 7 with N = 8.

Table 8: Numerical Results of example 3 for N = 16

index	xj	a_i	Uexact(xj)	Uapprox(xj)	Abs Error
1	3.1250e-02	-1.75e-11	3.13e-02	3.13e-02	0.00e+00
3	1.5625e-01	-7.75e-17	1.56e-01	1.56e-01	0.00e+00
5	2.8125e-01	-3.55e-16	2.81e-01	2.81e-01	5.55e-17
7	4.0625e-01	2.14e-14	4.06e-01	4.06e-01	0.00e+00
9	5.3125e-01	-7.15e-16	5.31e-01	5.31e-01	0.00e+00
11	6.5625e-01	-7.11e-16	6.56e-01	6.56e-01	0.00e+00
13	7.8125e-01	-5.80e-15	7.81e-01	7.81e-01	6.66e-16
15	9.0625e-01	3.18e-12	9.06e-01	9.06e-01	5.91e-14

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would decrease. Clearly to see this, we have a plot of these approximate solutions with the exact values as in Figure 8.

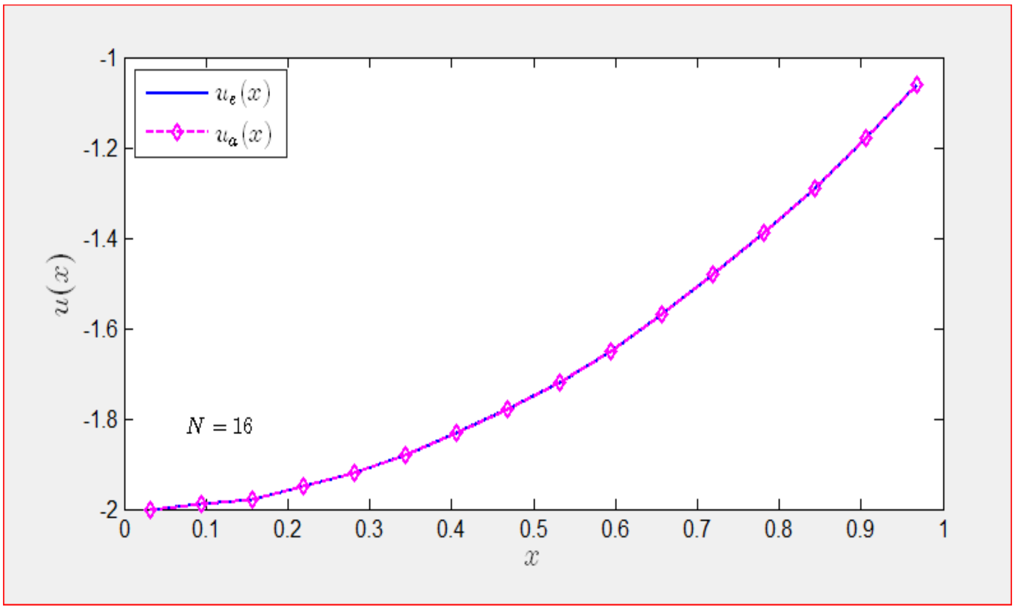

Figure 8. Numerical solutions of example 3 of table 8 with N = 16.

Table 9. Error analysis of example 3.

N	Maximum Error	Computation Time(sec)
4	4.5155e-26	0.079065
8	4.8844e-26	1.674471
16	1.2820e-25	3.775130
32	3.6313e-25	61.096492
64	3.6313e-25	1311.839093

From Table 9 we could conclude that for different value of N the computational time is increases and the error maximum (Emax) is decreases.

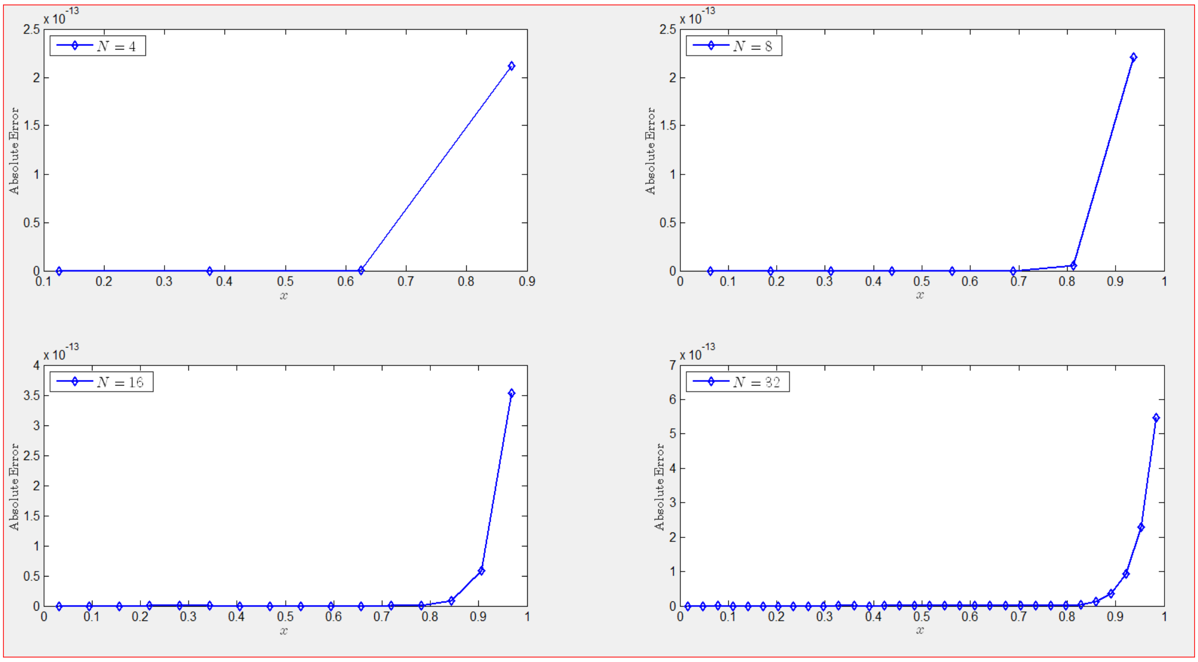

Figure 9: Absolute error of example 3 of above table for N =4, N=8, N=16. If we took more collocation points and we would get more accurate results. The collocation points are increases in each step lengths.

Example 5.4 Consider the following non-linear Volterra-Fredholm-Hammerstein integral equation given by.

$$u(x) = \cos(x) + \int_0^x (x-t)\cos(x-t)u(t)dt \qquad (6.59)$$

where $f(x) = \cos(x)$, $k_1(x,t) = 0$, $k_2(x,t) = (x-t)\cos(x-t)$ and the exact solution $u(x) = \frac{1}{3}\left[2\cos\sqrt{3}x + 1\right]$.

Step 1: Applying the Leibnitz rule to convert integral equation into differential equation.

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(x,t)dt = f(x,h(x))\frac{dh}{dx} - f\big(x,g(x)\big)\frac{dg}{dx} + \int_{g(x)}^{h(x)}\frac{\partial f}{\partial x}dt$$

If $g(x) = 0, h(x) = x$ and $\frac{d}{dx}\int_0^x f(x,t)dt = f(x,x) + \int_0^x \frac{\partial f}{\partial x}(x,t)dt$

Now $\quad u' = f' - \frac{d}{dx}\int_0^x (x-t)\cos(x-t)u(t)dt$,let $f_1 = \int_0^x (x-t)\cos(x-t)u(t)dt$,

$$\frac{d}{dx}f_1 = \int_0^x \frac{\partial f}{\partial x}[(x-t)\cos(x-t)]u(t)dt$$

$$= \int_0^x \cos(x-t)\,u(t)dt - \int_0^x (x-t)\sin(x)u(t)dt$$

$$u' = f' - \int_0^x \cos(x-t)\,u(t)dt - \int_0^x (x-t)\sin(x)u(t)dt \qquad (5.60)$$

$$u'' = f'' - \frac{d}{dx}\int_0^x \cos(x-t)\,u(t)dt - \frac{d}{dx}\int_0^x (x-t)\sin(x)u(t)dt$$

Let $f_2 = \int_0^x \cos(x-t)\,u(t)dt$ and $f_3 = \int_0^x (x-t)\sin(x)u(t)dt$

$$\frac{d}{dx}f_2 = \cos(x-x)u(x) + \int_0^x \frac{\partial f}{\partial x}\cos(x-t)\,u(t)dt$$

$$= u(x) - \int_0^x \sin(x)u(t)dt$$

$$\frac{d}{dx}f_3 = 0 + \int_0^x \frac{\partial f}{\partial x}\{(x-t)\sin(x-t)\}u(t)dt$$

$$= \int_0^x \sin(x-t)u(t)dt + \int_0^x (x-t)\cos(x-t)u(t)dt$$

$$u'' = f'' - u(x) + \int_0^x \sin(x-t)\,u(t)dt + \int_0^x \sin(x-t)u(t)dt$$

$$+ \int_0^x (x-t)\cos(x-t)u(t)dt$$

$$u'' = f'' - u(x) + 2\int_0^x \sin(x-t)\,u(t)dt + \int_0^x (x-t)\cos(x-t)\,u(t)dt \qquad (5.61)$$

But $\int_0^x (x-t)\cos(x-t)\,u(t)dt = \cos(x) - u(x)$

$$\mathrm{u}'' = \mathrm{f}'' - \mathrm{u(x)} + \cos(\mathrm{x}) - \mathrm{u(x)} + 2\int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$$

or

$$\mathrm{u}'' = \mathrm{f}'' - 2\mathrm{u(x)} + \cos(\mathrm{x}) + 2\int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt} \qquad (5.62)$$

$$u^{(3)} = f^{(3)} - \mathrm{s\,in}(x) - 2u' + 2\frac{\mathrm{d}}{\mathrm{dx}}\int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$$

Let $f_6 = \int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$

$$\frac{\mathrm{d}}{\mathrm{dx}}f_6 = \sin(\mathrm{x}-\mathrm{x})\,\mathrm{u(t)} + \int_0^{\mathrm{x}} \cos(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$$

$$= \int_0^{\mathrm{x}} \cos(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$$

$$u^{(3)} = f^{(3)} + \mathrm{s\,in}(x) - 2u' + 2\int_0^{\mathrm{x}} \cos(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt} \qquad (5.63)$$

$u^{(4)} = f^{(4)} - \cos(\mathrm{x}) - 2\mathrm{u}'' + \frac{\mathrm{d}}{\mathrm{dx}}\int_0^{\mathrm{x}} \cos(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$,let $f_6 = \int_0^{\mathrm{x}} \cos(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$

$$\frac{\mathrm{d}}{\mathrm{dx}}f_7 = \cos(x-x)\,u(x) - \int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$$

$$= \mathrm{u(x)} - \int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt}$$

$$u^{(4)} = f^{(4)} - \cos(\mathrm{x}) - 2\mathrm{u}'' + 2\mathrm{u(x)} - 2\int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt} \qquad (5.64)$$

Now we added (6.62) and (6.64)

$$\begin{cases} \mathrm{u}'' = \mathrm{f}'' - 2\mathrm{u(x)} + \cos(\mathrm{x}) + 2\int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt} \\ u^{(4)} = f^{(4)} - \cos(\mathrm{x}) - 2\mathrm{u}'' + 2\mathrm{u(x)} - 2\int_0^{\mathrm{x}} \sin(\mathrm{x}-\mathrm{t})\,\mathrm{u(t)dt} \end{cases}$$

$$u^{(4)} + \mathrm{u}'' = f^{(4)} + \mathrm{f}'' - 2\mathrm{u}''$$

$$u^{(4)} + 3\mathrm{u}'' - \left[f^{(4)} + \mathrm{f}''\right] = 0 \qquad (5.65)$$

(7) is not IVP yet

From (5.59) $u(0) = 1$ and from (5.60) $\mathrm{u}'(0) = 0$

From (5.61) $\mathrm{u}''(0) = \mathrm{f}''(0) - \mathrm{u}(0) = -2$ and

from (5.62) $u^{(3)}(0) = f^{(3)}(0) - \sin(0) - 2u'(0)$

$$= -2u'(0) = 0$$

$$\mathrm{u}^{(4)} + 3\mathrm{u}'' - \left[\mathrm{f}^{(4)} + \mathrm{f}''\right] = 0;\ u(0) = 1, \mathrm{u}'(0) = 0, \mathrm{u}''(0) - 2,\ u^{(3)}(0) = 0 \qquad (5.66)$$

(8) is IVP $f^{(0)} = \cos(x)$, $f^{(1)} = -\sin(x)$, $f^{(2)} = -\cos(x)$, $f^{(3)} = \sin(x)$, $f^{(4)} = \cos(x)$

$$f^{(4)} + f^{(2)} = \cos(x) - \cos(x) = 0$$

$$u^{(4)} + 3u'' = 0; u'_0 = 0, u''_0 = -2, u'''_0 = 0 \qquad (5.67)$$

So (5.67) is the final IVP

Collocation method using haar wavelets

$$u^{(4)} = \sum_{i=1}^{N} a_i \, h_i(x)$$

$$u^{(3)} - u^{(3)}_0 = \sum_{i=1}^{N} a_i \, p_{1,i}(x)$$

$$u^{(3)} = \sum_{i=1}^{N} a_i \, p_{1,i}(x)$$

$$u^{(2)} - u^{(2)}_0 = \sum a_i \, p_{2,i}(x)$$

$$u^{(2)} = -2 + \sum a_i \, p_{2,i}(x)$$

$$\sum_{i=1}^{N} a_i \, h_i(x) + 3 \sum a_i \, p_{2,i}(x) - 6 = 0$$

X collocates over [0,1] meshed up

$$x_j = \frac{j - 1/2}{N}, \, j = 1,2, \cdots, N$$

$$u'' = -2 + \sum_{i=1}^{N} a_i p_{2,i}(x)$$

$$u' = u'_0 - \int_0^x 2dt + \sum a_i \, p_{3,i}(x)$$

$$= u'_0 - 2x + \sum_{i=1}^{N} a_i p_{3,i}(x)$$

$$u(x) = u_0 - 2 \int x dx + \sum_{i=1}^{N} a_i p_{4,i}(x), \, u_0 = 1$$

$$u(x) = 1 - x^2 + \sum_{i=1}^{N} a_i p_{4,i}(x)$$

$u_a(x) = 1 - x^2 + \sum_{i=1}^{N} a_i p_{4,i}(x)$ is approximation solution for $u(x)$

Therefore, the numerical result for example 4 using MATLAB code D.

Table 10: Numerical Results of example 4 for N = 8

Index	xj	a_i	Uexact(xj)	Uapprox(xj)	Error
1	6.2500e-02	2.08e+00	9.96e-01	9.96e-01	1.23e-09
2	1.8750e-01	3.61e+00	9.65e-01	9.65e-01	1.04e-06
3	3.1250e-01	2.97e-01	9.05e-01	9.05e-01	2.12e-05
4	4.3750e-01	3.63e+00	8.18e-01	8.18e-01	1.48e-04
5	5.6250e-01	1.65e-02	7.08e-01	7.08e-01	6.01e-04
6	6.8750e-01	3.55e-01	5.81e-01	5.82e-01	1.73e-03
7	8.1250e-01	1.36e+00	4.42e-01	4.46e-01	3.85e-03
8	9.3750e-01	2.10e+00	2.98e-01	3.05e-01	6.88e-03

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would decrease. Clearly to see this, we have a plot of these approximate solutions with the exact values as in Figure 10.

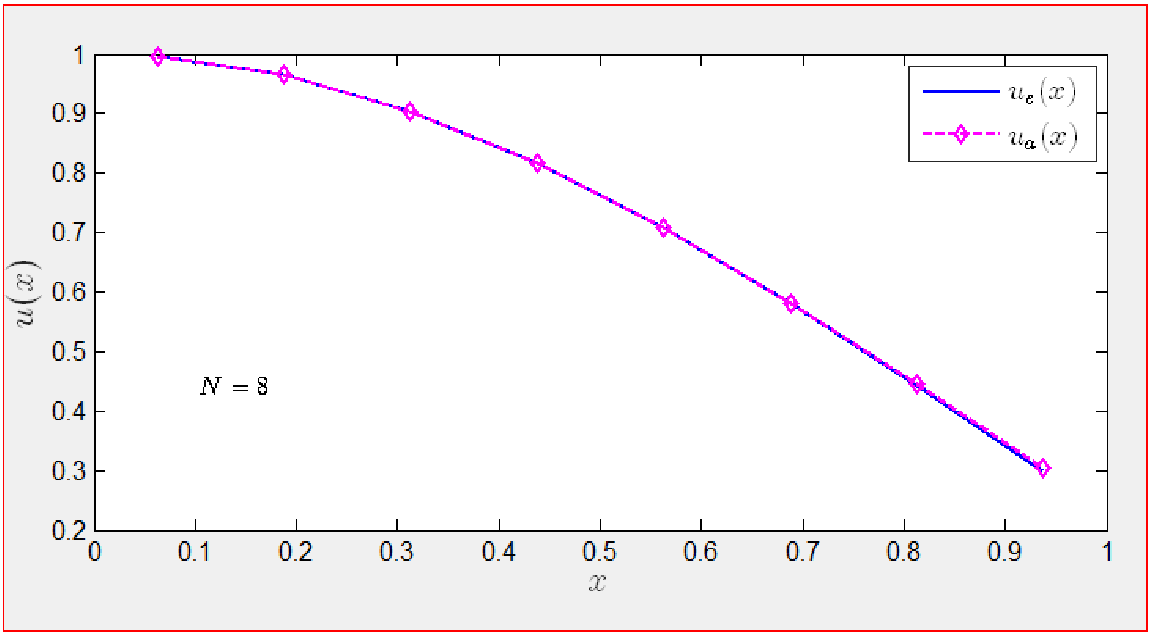

Figure 10. Numerical solutions of example 4 of table 10 with N = 8.

Table 11: Numerical Results of example 4 for N = 16

index	xj	a_i	Uexact(xj)	Uapprox(xj)	Error
1	3.1250e-02	2.05e+00	9.99e-01	9.99e-01	2.23e-11
3	1.5625e-01	3.06e-01	9.76e-01	9.76e-01	3.55e-07
5	2.8125e-01	1.89e-02	9.22e-01	9.22e-01	1.16e-05
7	4.0625e-01	1.37e+00	8.42e-01	8.42e-01	9.88e-05
9	5.3125e-01	1.03e-03	7.37e-01	7.38e-01	4.49e-04
11	6.5625e-01	1.03e-01	6.14e-01	6.15e-01	1.40e-03
13	7.8125e-01	5.32e-01	4.77e-01	4.81e-01	3.31e-03
15	9.0625e-01	1.05e+00	3.34e-01	3.40e-01	6.27e-03

As we have seen from this tabulated solution simply from their error values, we obtain accurate result by dividing into more sub interval that is as N increases the error would decrease. Clearly to see this, we have a plot of these approximate solutions with the exact values as in Figure 11.

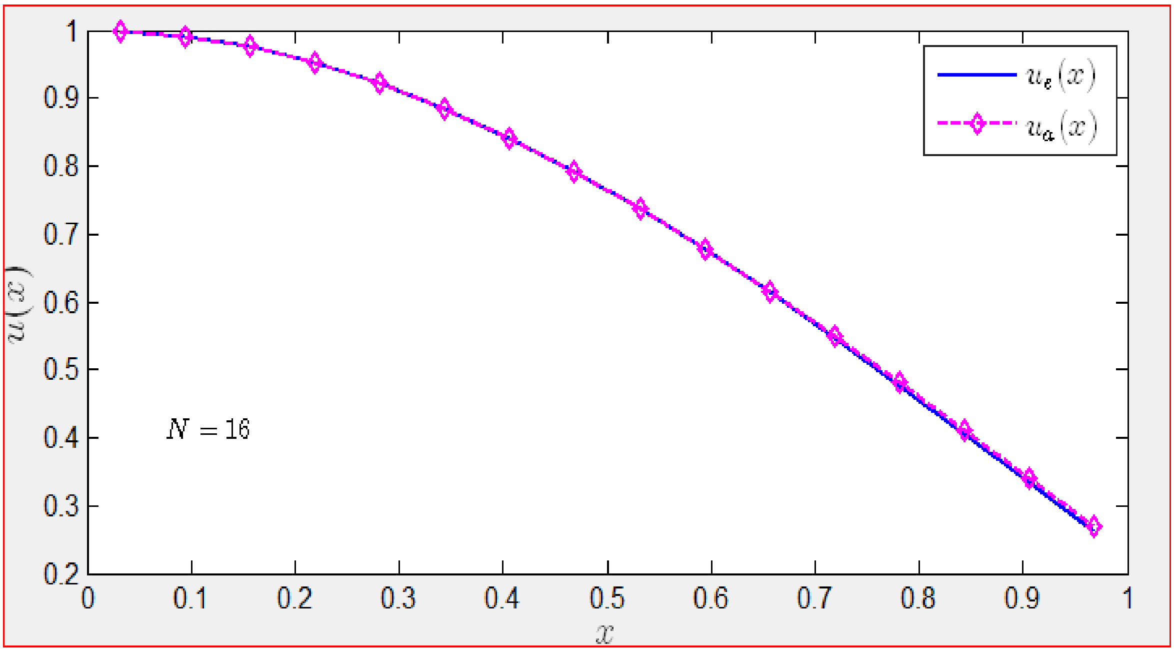

Figure 11. Numerical solutions of example 4 of table 11 with N = 16.

Table 12. Error analysis of example 4.

N	Maximum Error	Computation Time (sec)
4	2.2483e-05	0.410927
8	6.5509e-05	0.452118
16	0.0001433	4.688710
32	0.00029299	81.390101
64	0.00058922	1580.978389

From Table 12 we could conclude that as the dimension N increase the computational time increases and the maximum error (Emax) decreases.

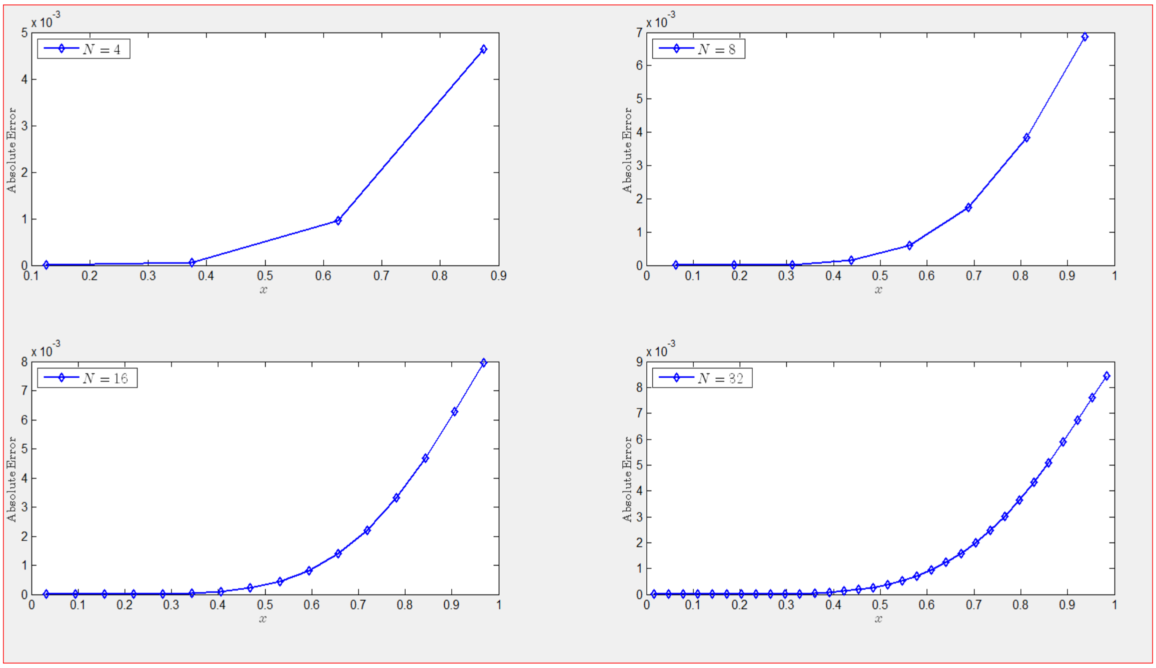

Figure 12: Absolute error of example 4 on table 12 for N =4, N=8, N=16.If we took more collocation points and we would get more accurate results. The collocation points are increases in each step lengths.

6. SUMMARY AND CONCLUSION

6.1. SUMMARY

The main objective this project was to solve non-linear Volterra-Fredholm-Hammerstein integral equations, which observed in many areas that arise in the theory of parabolic boundary value problems, engineering; various mathematical physics, and theory of elasticity etc. are treated numerically by HWCM. It is difficult to get solutions for plenty of them because of their variety of categories and the behaviors they usually depict. In this project, we have seen how non-linear integral equations, which can not be solved analytically, can be approximated by using family of numerical method (haar wavelet collocation method). HWCM was used to solve non-linear integral equations by reducing the nonlinear integral equations to a nonlinear system of algebraic equations. Haar wavelet Collocation method based on Leibnitz rule was applied to obtain the numerical solution of non-linear Volterra-Fredholm-Hammerstein integral equation of the second kind.

Haar wavelet function and its Operational matrix were employed to solve the resultant integral equations utilized to transform the nonlinear integral equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown coefficients. The solutions exhibits non-linear integral equations depending of the equation reduced problem. The problems and the exact solution are taken from Ordokhani and Razzaghi in (2011). But the numerical solution was obtained with the MATLAB code of the numerical scheme of this project. The numerical results for four test examples demonstrated the efficiency of the method and the computational costs for those method are analyzed for several examples. The numerical results are taken by using MATLAB coding and the numerical results are obtained by the proposed method have been demonstrated in tables and figures.

6.2. CONCLUSION

This project work provided a thorough examination of existing haar wavelet collocation method approaches for non-linear Volterra-Fredholm-Hammerstein integral equations as well as extensions of the method in different directions. Haar wavelet collocation method was formulated with in a common abstract framework enabling their efficient application to complex problems. Haar Wavelet Collocation method was applicable in science and engineering, since they are readily evaluates different non-linear integral equations. Haar Wavelet Collocation method used to reduce the nonlinear integral equations to a nonlinear system of algebraic equations by applied Leibnitz rule. Haar wavelet operational matrix utilized to transform the nonlinear integral equation to a matrix equation which corresponds to a system of non-linear algebraic equations with unknown coefficients.From the results of the study as the values of the N increases and the error were increases and computational costs were increases and to obtain more accurate results for those examples. Furthermore, the proposed method could be implemented in MATLAB code. Lastly, from the result, we obtain haar wavelet collocation method is more accurate than proposed method.

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