

**NUMERICAL SOLUTION OF SECOND ORDER SINGULARLY  
PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS**

**M.Sc. PROJECT**

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Difference Equations**

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MASTER OF SCIENCE IN MATHEMATICS  
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**By**

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As MSc project research advisors, we hereby certify that we have read and evaluated this project entitled “**Numerical Solution of Second Order Singularly Perturbed Differential-Difference Equations**” prepared under our guidance by Werku Debele. We recommended that it be submitted as fulfilling the project requirement.

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Final approval and acceptance of the Project is contingent upon the submission of its final copy to the council of graduate studies (CGS) through the candidate’s department or school graduate committee (DGC or SGC).

## **DEDICATION**

This project is dedicated to my beloved Keneni Gobu and my families who bring me up to this level, nursing me with affection and their giving love for my work and successes in my life.

## STATEMENT OF THE AUTHOR

By my signature below, I declare and affirm that this project is my own work. I have followed all the ethical and technical principles of scholarship in the preparation, and compilation of this project. Any scholarly matter that is included in the project has been given recognition through citation.

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## **ABBREVIATIONS**

BVP	Boundary Value Problem
DDE	Differential Difference Equation
DE	Differential Equation
FDM	Finite Difference Method
ODE	Ordinary Differential Equation
SP	Singularly Perturbed
SPDDE	Singularly Perturbed Differential-Difference Equation

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# Numerical Solution of Second Order Singularly Perturbed Differential-Difference Equations

## ABSTRACT

*The purpose of this study is to investigate the numerical solution of second order singularly perturbed differential-difference equations by using second order finite difference methods. Taylor series expansion is used to expand the terms containing the shift parameters and Finite Difference Method is applied to discretize the given equations to provide the derivation of the numerical scheme. The second order singularly perturbed differential difference equations is replaced by an asymptotically equivalent singularly perturbed boundary value problem. A fitting factor is introduced in the finite difference scheme which takes care of the rapid changes that occur in the boundary layer and is obtained from the theory of singular perturbations. The convergence of the numerical scheme is discussed. Thomas Algorithm is used to solve the tri-diagonal system and its stability investigated. To validate the applicability of the method, four test examples have been solved by taking different values for the delay parameter  $\delta$ , advanced parameter  $\eta$  and the perturbation parameter  $\varepsilon$ . The MATLAB code is developed and all the computations are performed using MATLAB.*

**Key words:** Boundary layer, Differential- Difference Equations, Finite Difference Method and Singular Perturbations.

# 1. INTRODUCTION

## 1.1. Background of the Study

Numerical analysis is a branch of Mathematics that deals with devising efficient methods for obtaining numerical solutions to difficult mathematical problems. Most of the mathematical problems that arise in science and engineering are very hard and sometime impossible to solve explicitly. Thus, an approximation to a difficult mathematical problem is very important to make it easier to solve (Baskar, 2010). In numerical analysis, a numerical method is a mathematical tool designed to solve numerical problems.

In mathematics, a differential equation is an equation in which one or more of the derivatives of some function appear. The order of a differential equation is the highest order derivative occurring. Differential equations can describe nearly all systems undergoing change. They are ubiquitous in science and engineering as well as economics, social science, biology, business, health care, etc. Many mathematicians have studied the nature of these equations for hundreds of years and there are many well-developed solution techniques.

Differential-difference equations (equations containing negative and positive shifts of the unknown function and its derivatives) and their systems occur in many fields of science and engineering. Many phenomena in economics, biosciences, etc., can be modeled as systems of differential-difference equations, therefore it is necessary to solve these equations, numerically. There are several numerical methods for obtaining the solution of differential-difference equations (Dadkhah Tirani *et al.*, 2015). In recent years, the studies of differential-difference equations, i.e., equations containing shifts (negative and positive shifts) of the unknown function and its derivatives have developed very rapidly and intensively (Bainov *et al.*, 2000; Cao and Wang, 2004; El'sgol'ts and Norkin, 1973). The problems in which the highest order derivative term is multiplied by a small positive parameter  $\varepsilon$  ( $0 < \varepsilon \ll 1$ ) are known to be perturbed problems and the parameter  $\varepsilon$  is known as perturbation parameter. Depending on the solution behavior of the problem, in the limiting case when perturbation parameter goes to zero, such types of problems are classified into two, namely:-

- (i) Regularly perturbed and

(ii) Singularly perturbed.

If the solution of the original problem tends to the solution of the reduced problem (i.e., the problem which is obtained by putting  $\varepsilon = 0$  in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed, otherwise it is known as singularly perturbed (Phaneendra *et al.*, 2014). A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term. In recent papers the terms negative or left shift and positive or right shift have been used for delay and advance respectively.

Singular perturbation is a small parameter and it is a maturing area of mathematics with a fairly long history and a strong promise for continued important applications in science and engineering; for instance, in fluid mechanics, fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, magneto hydrodynamics, reaction diffusion processes, and many other problems of fluid motion. The smoothness of the solutions of such singularly perturbed differential- difference equations deteriorates when the parameter tends to zero.

The second order Singularly Perturbed Differential-Difference Equations (SPDDEs) has the form:-

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x - \delta) + c(x)y(x) + d(x)y(x + \eta) = f(x), \forall x \in [0,1] \quad (1)$$

under the boundary conditions

$$y(x) = \varphi(x), \quad -\delta \leq x \leq 0 \quad (2)$$

$$y(x) = \gamma(x), \quad 1 \leq x \leq 1 + \eta \quad (3)$$

where  $\varepsilon$  is small parameter  $0 < \varepsilon \ll 1$ ,  $\delta$  and  $\eta$  are also small shifting parameters,  $0 < \delta \ll 1$  and  $0 < \eta \ll 1$ ;  $a(x), b(x), c(x), d(x), f(x), \varphi(x)$  and  $\gamma(x)$  are smooth functions (has continuous derivatives up to some desired order over domain  $[0,1]$ ).

For a function  $y(x)$  to be a smooth solution of the problem (1), it must satisfy equation (1) with the given boundary conditions (2) and (3), be continuous on  $[0,1]$  and continuously differentiable on  $(0,1)$ .

This project is concerned with numerically solving some classes of second order Singularly Perturbed Differential-Difference Equations using the second order Finite Difference Method (FDM).

## 1.2. Statement of the problem

Singularly perturbed differential-difference equations (SPDDEs) arise very frequently in the mathematical modeling of real life situations in science and engineering (Shu, 2012). The differential-difference (Delay and advance differential) equation play an important role in the mathematical modeling of various practical phenomena in the biosciences and control theory (Lange and Miura, 1982). Gemechis and Reddy (2012) presented Fitted-Modified Upwind Finite Difference Method for Solving Singularly Perturbed Differential Difference Equations or singularly perturbed boundary value problems with delay  $\delta$  and advance  $\eta$  parameters that are sufficiently small having the boundary layer at one end (left or right). In this study authors shows stability of the algorithm but did not discuss the convergence analysis.

Phaneendra *et al.* (2014) presented a numerical finite difference approach to solve the boundary-value problem for singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term. In this method, they first approximate the shifted term by Taylor series and applied a fourth order finite difference scheme, provided shifts are of  $o(\epsilon)$  that means the delay is of small order of the singular perturbation parameter. Sirisha and Reddy (2014) discussed the numerical solution of singularly perturbed differential-difference equations of mixed type exhibiting dual layer behavior. In this study authors discussed the error analysis and convergence of the scheme but did not discuss the stability of the algorithm.

However, in this project the study tried to numerically solve second order Singularly Perturbed Differential-Difference Equations (SPDDEs) of the type (1) with both negative and positive shifts whose solutions exhibits layer behavior on the both left-end and right-end of the interval with the convergence analysis and stability analysis investigated together. For solving the above equation, we used the Taylor series expansion to expand the terms  $y(x - \delta)$  and  $y(x + \eta)$  and then applied second order Finite Difference Method to developed the numerical scheme.

### 1.3. Objectives

The general objective of this project is to study the numerical solution of second order singularly perturbed differential-difference equations by using second order finite difference methods.

The specific objectives:

- ❖ To provide derivation of the numerical scheme of the proposed finite difference method.
- ❖ To study the stability of the numerical scheme.
- ❖ To investigate the convergence of the numerical scheme for uniform mesh size  $h$ .
- ❖ To demonstrate the efficiency of the proposed finite difference method using different test problems with different values of the shifting (advance and delay) and the perturbation parameters.

## 2. REVIEW OF RELATED LITERATURE

The term "singular perturbation" was coined in the 1940s by Kurt Otto Friedrichs and Wolfgang R. Wasow (1981). The study of many theoretical and applied problems in science and technology leads to boundary value problems for singularly perturbed differential equations that have a multi-scale character. However, most of the problems cannot be completely solved by analytic techniques.

A perturbed problem whose solution can be approximated on the whole problem domain, whether space or time, by a single asymptotic expansion has a regular perturbation. Most often in applications, an acceptable approximation to a regularly perturbed problem is found by simply replacing the small parameter by zero everywhere in the problem statement. This corresponds to taking only the first term of the expansion, yielding an approximation that converges, perhaps slowly, to the true solution as decreases. The solution to a singularly perturbed problem cannot be approximated in this way. Singular perturbation problems appear in various applications ranging from physical, chemical, and biological to engineering sciences. Typically, these problems are characterized by a small positive parameter multiplying the highest order derivative terms. In general, boundary layers or interior layers occur in their solutions (Zhongdi *et al.*, 2018).

In order to solve second order Singularly Perturbed Differential-Difference Equation, plenty of numerical methods were investigated during the past years. Bellman *et al.* (1972) introduced the Differential Quadrature Method (DQM) in the early 1970s and, since then, the technique has been successfully employed in finding the solutions of many problems in applied and physical sciences (Shu, 2012). The basic idea of differential quadrature method is that the derivative of a function with respect to a space variable at a given point is approximated as a weighted linear sum of the functional values at all discrete points in the domain of that variable. Prasad and Reddy (2012) proposed the Differential Quadrature Method (DQM) for finding the numerical solution of boundary-value problems for a singularly perturbed differential-difference equation of mixed type, i.e., containing both terms having a negative shift and terms having a positive shift.

Bellman *et al.* (1972) proposed two approaches to compute the weighting coefficients. The first approach solves an algebraic equation system and the second approach uses a simple algebraic formulation, but with the coordinates of grid points chosen as the roots of the shifted Legendre polynomial. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill- conditioned. Thus it is very difficult to compute the weighting coefficients for a large number of grid points. To improve the Bellman's approaches in computing the weighting coefficients, many attempts have been made by researchers. One of the most useful approaches is the one introduced by Quan and Chang, (1989).

After that Shu, (2012) generalized approach based on the high order polynomial approximation and linear vector space analysis, made available in the literature. This generalized approach computes the weighting coefficients of the first order derivative by a simple algebraic formulation without any restriction on choice of grid points, and the weighting coefficients of second and higher order derivatives by a recurrence relationship. Gemechis and Reddy (2012) presented Fitted-Modified Upwind Finite Difference Method for Solving Singularly Perturbed Differential Difference Equations or singularly perturbed boundary value problems with delay  $\delta$  and advance  $\eta$  parameters that are sufficiently small having the boundary layer at one end (left or right) and the stability of the algorithm is also considered.

Awoke and Reddy (2013) presented a parameter fitted scheme to solve singularly perturbed delay differential equations of second order with left and right boundary. Sirisha and Reddy (2014) discussed the numerical solution of singularly perturbed differential-difference equations of mixed type exhibiting dual layer behavior and the error bound and convergence of the method have also been established in the study. Sirisha and Reddy (2014) presented a fitted upwind difference scheme for solving singularly perturbed differential-difference equations with negative shift whose solutions exhibits boundary layer behavior. In the study the scheme is to be repeated for different choices of the delay parameter  $\delta$  and perturbation parameter  $\varepsilon$ . The choice of  $\delta$  is not unique, but can assume any number of values satisfying the condition,  $0 < \delta \ll 1$  and  $0 < \varepsilon' \ll 1$  such that  $\varepsilon' = \varepsilon - \delta\alpha(x)$ ,  $\forall x \in [0, 1]$ .

It can be observed from the tables that the proposed method approximates the exact solution very well and also produces good and/or consistent results which are in support of the theory for different values of  $\varepsilon$  and  $\delta$  for problems without exact solutions which in turn implies the efficiency of the method. The delay parameter  $\delta$  affects both the boundary layer solutions (left and right) in similar fashion but reversely. That is as  $\delta$  increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases.

Sirisha and Reddy (2015) presented an exponentially fitted initial value technique for solving singularly perturbed differential-difference equations with delay as well as advance terms whose solutions exhibit boundary layer on one (left/right) of the interval. Theoretical convergence of the scheme has also been investigated. Sirisha *et al.* (2016) proposed a mixed finite difference method to solve singularly perturbed differential-difference equations with mixed shifts, solutions of which exhibit boundary layer behavior at the left end of the interval using domain decomposition. A terminal boundary point is introduced into the domain, to decompose it into inner and outer regions.

Kumara Swamy *et al.* (2016) presented Galerkin method to solve singularly perturbed differential-difference equations with delay and advanced shifts using fitting factor. In the numerical treatment of such type of problems, Taylor's approximation is used to tackle the terms containing small shifts. A fitting factor in the Galerkin scheme is introduced which takes care of the rapid changes that occur in the boundary layer. This fitting factor is obtained from the asymptotic solution of singular perturbations. Thomas algorithm is used to solve the tri-diagonal system of the fitted Galerkin method.

### 3. MATERIALS AND METHODS

This chapter outlines the methods and materials used to find the numerical solution of second order singularly perturbed differential-difference equations with small shifts of mixed type and materials used in the study. Sources in the web and libraries were used to collect all the pieces of information about numerical solution of second order singularly perturbed differential-differences. Specifically,

- ❖ Books, related studies from internet services and any available material that support the study were addressed to gather information about the singularly perturbed differential-difference equation with small shifts of mixed type.
- ❖ Taylor series expansion was used to expand the terms containing the shift parameters.
- ❖ A fitting factor in the Finite Difference scheme was introduced which takes care of the rapid changes that occur in the boundary layer. This fitting factor was obtained from the asymptotic solution of singular perturbations. Thomas algorithm was used to solve the tri-diagonal system.
- ❖ MATLAB code were developed and used to make easy the computations by the methods and graphs were plotted using the programs.

## 4. PRELIMINARIES

In this chapter, we deal with definitions, theorems and concepts which are important for the study of solving singularly perturbed differential-difference equations with small shifts of mixed type by using finite difference method.

### Basic Concepts and Definitions

#### Definition 4.1: Singularly Perturbed Differential-Difference Equations

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term (Lange and Miura, 1982). Singularly Perturbed Differential-difference equations (SPDDEs), also called as a class of functional differential equations, are mathematical models of a number of real phenomenon. Singularly perturbed differential-difference equations is said to be singularly perturbed delay differential equation if it contains only negative shift and singularly perturbed differential-difference equation if it contains both negative and positive shifts (often referred as “delay and advanced” parameters respectively) (phaneendra *et al.*, 2014).

**Definition 4.2: (Perturbation theory):** is a subject which studies the effect of small parameter in the mathematical model problems in differential equations.

**Definition 4.3: (Theory of singular perturbation):** concerns the studies of problems featuring a parameter for which the solutions of the problem at a limiting value of the parameter are different in character from the limit of the solutions of the general problem; namely, the limit is singular. In contrast, for regular perturbation problems, the solutions of the general problem converge to the solutions of the limit-problem as the parameter approaches the limit-value.

**Definition 4.4: (Singular perturbation problem):** is a problem containing a small parameter that cannot be approximated by setting the parameter value to zero.

**Definition 4.5: (Taylor series):** Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. A Taylor series of a function  $f(x)$  about a point  $x = a$  is a power series representation of

$f(x)$  developed so that all the derivatives of  $f(x)$  at a match all the derivatives of the power series. Without worrying about convergence here, we have

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots$$

**Definition 4.6:** A finite difference is a mathematical expression of the form  $f(x + b) - f(x + a)$ . If a finite difference is divided by  $b - a$ , one gets a difference quotient. Then approximation of derivatives by finite differences plays a central role in finite difference methods for the numerical solution of differential equations, especially boundary value problems. By using the Taylor series the second order central difference is given by

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h}$$

$$f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}$$

**Definition 4.7: (Absolute error):** is the absolute difference between the true value and the approximations value.

**Definition 4.8:** The order of convergence is one of the primary ways to estimate the actual rate of convergence the speed at which the errors go to zero. Typically the order of convergence measures the asymptotic behavior of convergence, often up to constants. For example, Newton's method is said to have quadratic convergence, so the method has order 2.

**Definition 4.9: (Tri-diagonal matrix algorithm - TDMA (Thomas algorithm)):**

A tri-diagonal system may be written as

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i, i = 1, 2, 3, \dots, N \text{ and } a_1 = c_n = 0.$$

In matrix form, this system is written as

$$\begin{bmatrix} b_1 & c_1 & \dots & \dots & \dots & 0 \\ a_2 & b_2 & c_2 & \dots & \dots & \vdots \\ \vdots & a_3 & b_3 & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \ddots & c_{n-1} \\ 0 & \dots & \dots & \dots & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{bmatrix}$$

## Variants

In some situations, particularly those involving periodic boundary conditions, a slightly perturbed form of the tri-diagonal system may need to be solved:

$$a_1x_n + b_1x_1 + c_1x_2 = d_1,$$

$$a_ix_{i-1} + b_ix_i + c_ix_{i+1} = d_i, i = 2, 3, \dots, n-1,$$

$$a_nx_{n-1} + b_nx_n + c_nx_1 = d_n.$$

In matrix form, this is

$$\begin{bmatrix} b_1 & c_1 & \cdots & a_1 \\ a_2 & b_2 & c_2 & \vdots \\ \vdots & a_3 & b_3 & c_3 \\ \vdots & \ddots & \ddots & c_{n-1} \\ c_n & \cdots & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}.$$

## Algorithm

The TDMA (Thomas Algorithm) is based on the Gaussian elimination procedure and consist of two parts: a forward elimination phase and a backward substitution phase.

### Forward elimination phase

for  $k \geq 2$  step until  $n$  do

$$m = \frac{a_k}{b_{k-1}}$$

$$a_k = a_k - mb_{k-1}$$

$$b_k = b_k - mc_{k-1}$$

$$d_k = d_k - md_{k-1}$$

end loop ( $k$ )

### Backward substitution phase

$$x_n = \frac{d_n}{b_n}$$

for  $k = n-1$  step down until 1 do

$$x_k = \frac{d_k - c_k x_{k+1}}{b_k}$$

end loop ( $k$ )

**Definition 4.10: (Stability):** A numerical method is said to be stable if it does not magnify the errors that appear in the course of the numerical solution process.

**Definition 4.11: (Convergence):** A numerical method is said to be convergent if the solution of the discretized equations tends to the exact solution of the differential equation as the grid spacing tends to zero.

## 5. NUMERICAL SOLUTION OF SECOND ORDER SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS

### 5.1. General Description of the Problem

Consider a second order linear singularly perturbed differential-difference equation with small delay as well as advance parameter of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x - \delta) + c(x)y(x) + d(x)y(x + \eta) = f(x), \quad \forall x \in [0,1] \quad (1)$$

subject to the interval and boundary conditions

$$y(x) = \varphi(x), \text{ on } -\delta \leq x \leq 0 \quad (2)$$

$$y(x) = \gamma(x), \text{ on } 1 \leq x \leq 1 + \eta \quad (3)$$

where  $a(x), b(x), c(x), d(x), \varphi(x), \gamma(x)$  are bounded and continuously differentiable functions on  $(0,1)$ ,  $0 < \varepsilon \ll 1$  is the singular perturbation parameter; and  $0 < \delta \ll 1$  and  $0 < \eta \ll 1$  are the delay and the advance parameters respectively.

In general, the solution of (1) – (3) exhibits boundary layer behavior at one end of the interval  $[0,1]$  depending on the sign of the coefficient of the first derivative of equation (6) given below.

By using Taylor series expansion in the neighborhood of the point  $x$ , we have

$$y(x - \delta) \approx y(x) - \delta y'(x) \quad (4)$$

$$y(x + \eta) \approx y(x) + \eta y'(x) \quad (5)$$

Using equations (4) and (5) in (1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \quad (6)$$

$$y(0) = \varphi(0) = \varphi_0 \quad (7)$$

$$y(1) = \gamma(1) = \gamma_1 \quad (8)$$

where 
$$p(x) = a(x) + \eta d(x) - \delta b(x) \quad (9)$$

and 
$$q(x) = b(x) + c(x) + d(x) \quad (10)$$

The transition from equation(1) to (6) is admitted, because of the condition that  $0 < \delta \ll 1$  and  $0 < \eta \ll 1$  are sufficiently small. This replacement is significant from the computational point of view.

## 5.2. Numerical Scheme

### 5.2.1. Left End Boundary Layer Problems

Let  $0 = x_0, x_1, \dots, x_N = 1$  be a decomposition of the considered interval  $[0,1]$  into  $N$  equal subintervals with constant mesh size  $h = \frac{1}{N}$ .

Then we have the nodes  $x_i = ih$ , for  $i = 0, 1, \dots, N$ . Assume that  $p(x), q(x)$  and  $f(x)$  are sufficiently continuously differentiable functions in  $[0,1]$ .

If 
$$p(x) = a(x) + \eta d(x) - \delta b(x) \geq M > 0$$

$$q(x) = a(x) + c(x) + d(x) \leq 0, \text{ through the interval } [0,1],$$

where  $M$  is some positive constant. Under these assumptions, (6) has a unique solution  $y(x)$  which exhibits a boundary layer of width  $O(\varepsilon)$  on the left side ( $x=0$ ) of the underlying interval.

**Lemma 1:** (Doolan *et al.*, 1980 and O'Malley, 1974) Let  $y(x) = y_0 + z_0$  be the zeroth-order asymptotic approximation to the solution of (6), where  $y_0$  represents the zeroth-order approximate outer solution (i.e., the solution of the reduced problem of (6)) and  $z_0$  represents the zeroth-order approximate solution in the boundary layer region of (6).

Then, for a fixed positive integer  $i$ ,

$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\varphi(0) - y_0(0))e^{-(p(0) - \frac{\varepsilon q(0)}{p(0)})i\rho} \quad (11)$$

where 
$$\rho = \frac{h}{\varepsilon}$$

Proof: let  $y_0(x)$  be the solution of the reduced problem of (6)

$$p(x)y_0'(x) + q(x)y_0(x) = f(x), \quad y_0(1) = \gamma(1) \quad (12).$$

and  $z_0(t)$  is the solution of the boundary value problem of (6) (O'Malley, 1974)

$$z_0''(t) + p(0)z_0'(t) = 0, \quad z_0(0) = \varphi(0) - y_0(0), \quad z_0(\infty) = 0 \quad (13)$$

where  $t = \frac{x}{\varepsilon}$ . From the theory of singular perturbations, it is known that the solution of equations (6)-(8) is of the form (O' Malley, 1974)

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}(\varphi(0) - y_0(0))e^{-\int_0^x \left(\frac{p(0)}{\varepsilon} - \frac{q(x)}{p(x)}\right) dx} + o(\varepsilon) \quad (14)$$

As we are considering the differential equations on sufficiently small subintervals or about the point '0', the coefficients could be assumed to be locally constant. Hence,

$$y(x) = y_0(x) + (\varphi(0) - y_0(0))e^{-\int_0^x \left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right) dx} + o(\varepsilon) \quad (15)$$

So, at the nodal points, we have

$$y(x_i) = y_0(x_i) + (\varphi(0) - y_0(0))e^{-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right)x_i} + o(\varepsilon), \quad i = 0, 1, \dots, N.$$

that is, 
$$y(ih) = y_0(ih) + (\varphi(0) - y_0(0))e^{-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right)ih} + o(\varepsilon)$$

Therefore 
$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\varphi(0) - y_0(0))e^{-(p(0) - \frac{\varepsilon q(0)}{p(0)})i\rho} \quad (16)$$

where 
$$\rho = \frac{h}{\varepsilon}$$

From equation (6) at  $x = x_i$ , we have:

$$\varepsilon y''(x_i) + p(x_i)y'(x_i) + q(x_i)y(x_i) = f(x_i) \quad (17)$$

and using the Second order central differences:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)}(\xi_1) + R_1 \quad (18)$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i'''(\xi_2) + R_2 \quad (19)$$

Here  $R_1 = -\frac{2h^4 y^{(6)}(\xi_1)}{6!}$  and

$$R_2 = -\frac{2h^4 y^{(5)}(\xi_2)}{5!} \text{ for } \xi_1, \xi_2 \in [x_{i-1}, x_i].$$

Now, from (19) and (18) in (17) we have:

$$\frac{\varepsilon}{h^2} [y_{i+1} - 2y_i + y_{i-1}] + \frac{p_i}{2h} [y_{i+1} - y_{i-1}] - \frac{h^2 p_i}{6} y_i''' + q_i y_i = f_i + R \quad (20)$$

where  $R = -\varepsilon R_1 - p_i R_2 + \frac{\varepsilon h^2 y^{(4)}(\xi_1)}{12}$

$$p(x_i) = p_i, q(x_i) = q_i, f(x_i) = f_i, y(x_i) = y_i$$

From (6) we have

$$\varepsilon y_i'' = f_i - p_i y_i' - q_i y_i \quad (21)$$

Differentiating both sides of equation (21) we have

$$y_i''' = \frac{1}{\varepsilon} (f_i' - p_i y_i'' - (p_i' + q_i) y_i' - q_i' y_i)$$

then substituting into (20) we have:

$$\begin{aligned} \frac{\varepsilon}{h^2} [y_{i+1} - 2y_i + y_{i-1}] + \frac{p_i}{2h} [y_{i+1} - y_{i-1}] + \frac{h^2 p_i}{6\varepsilon} (p_i y_i'' + (p_i' + q_i) y_i' + q_i' y_i) + q_i y_i = f_i + \\ \frac{h^2 p_i}{6\varepsilon} f_i' + R \end{aligned} \quad (22)$$

Now, we approximate the converted error term, which has the stabilizing effect, in equation (22) by using the central difference formula for  $y_i'$  and  $y_i''$  from equations (18) and (19), we obtain the Second order central difference scheme:

$$\begin{aligned} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) \left( \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \left( p_i + \frac{h^2 p_i}{6\varepsilon} (p_i' + q_i) \right) \left( \frac{y_{i+1} - y_{i-1}}{2h} \right) + \left( q_i + \frac{h^2 p_i q_i'}{6\varepsilon} \right) y_i \\ = f_i + \frac{h^2 p_i}{6\varepsilon} f_i' + \tau_i \end{aligned} \quad (23)$$

where  $\tau_i = \left( \frac{h^2 \varepsilon}{12} + \frac{p_i^2 h^4}{72\varepsilon} \right) y_i^{(4)} + \frac{p_i (p_i' + q_i) h^4}{36\varepsilon} y_i''' - \varepsilon R_1 - p_i R_2$  is the local truncation error.

Now introducing a fitting factor  $\sigma$  into equation (23) we obtain

$$\sigma \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) \left( \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \left( p_i + \frac{h^2 p_i}{6\varepsilon} (p_i' + q_i) \right) \left( \frac{y_{i+1} - y_{i-1}}{2h} \right) + \left( q_i + \frac{h^2 p_i q_i'}{6\varepsilon} \right) (y_i) = f_i + \frac{h^2 p_i}{6\varepsilon} f_i' + \tau_i \quad (24)$$

$y_0 = \varphi_0$ ,  $y_N = \gamma_1$ . Here  $\sigma$  is a fitting factor which is to be determined in such a way that the solution of (24) with boundary conditions converges uniformly to the solution of equations (6) – (8) which is in turn a good approximation to the solution of equations (1) – (3).

Multiplying (24) by  $h$  and taking limit as  $h \rightarrow 0$  we obtain  $\sigma$

$$\lim_{h \rightarrow 0} \frac{\sigma}{\rho} \left( 1 + \frac{\rho^2 p^2(ih)}{6} \right) (y(ih + h) - 2y(ih) + y(ih - h)) + \lim_{h \rightarrow 0} \frac{1}{2} p(ih) \left( 1 + \frac{\rho h}{6} \right) (p'(ih) + q(ih)) (y(ih + h) - y(ih - h)) = 0 \quad (25)$$

where  $\rho = \frac{h}{\varepsilon}$  and  $f_i + \frac{h^2 p_i}{6\varepsilon} f_i' - (q_i + \frac{h^2 p_i q_i'}{6\varepsilon}) y_i$  is bounded.

Now, approximating the solution  $y(x)$  by zeroth-order asymptotic approximation  $y_0(x)$  and using Lemma1, we have

$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\varphi(0) - y_0(0)) e^{-\left(p(0) - \frac{\varepsilon q(0)}{p(0)}\right) i \rho} \quad (26a)$$

$$\lim_{h \rightarrow 0} y(ih + h) = y_0(0) + (\varphi(0) - y_0(0)) e^{-\left(p(0) - \frac{\varepsilon q(0)}{p(0)}\right) i \rho} e^{-\left(p(0) - \frac{\varepsilon q(0)}{p(0)}\right) \rho} \quad (26b)$$

$$\lim_{h \rightarrow 0} y(ih - h) = y_0(0) + (\varphi(0) - y_0(0)) e^{-\left(p(0) - \frac{\varepsilon q(0)}{p(0)}\right) i \rho} e^{\left(p(0) - \frac{\varepsilon q(0)}{p(0)}\right) \rho} \quad (26c)$$

By substituting equations (26a) – (26c) in (25), we get the fitting factor as:

$$\sigma = \left( \frac{3\rho p(0)}{6 + \rho^2 p^2(0)} \right) \coth \left[ \left( \frac{p^2(0) - \varepsilon q(0)}{p(0)} \right) \frac{\rho}{2} \right] \quad (27)$$

Finally, making use of equation (24) and equation (27), we get the three term recurrence relation of form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, 3, \dots, N - 1 \quad (28)$$

where  $E_i = \frac{\sigma}{h^2} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) - \frac{p_i}{2h} \left( 1 + \frac{h^2}{6\varepsilon} (p_i' + q_i) \right)$

$$F_i = \frac{2\sigma}{h^2} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) - \left( q_i + \frac{h^2 p_i q_i'}{6\varepsilon} \right)$$

$$G_i = \frac{\sigma}{h^2} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) + \frac{p_i}{2h} \left( 1 + \frac{h^2}{6\varepsilon} (p_i' + q_i) \right)$$

$$H_i = f_i + \frac{h^2 p_i f_i'}{6\varepsilon}$$

The difference equation (28) forms a tri-diagonal system of  $N - 1$  equations with  $N + 1$  unknowns  $y_0, y_1, \dots, y_N$ . The  $N - 1$  equations together with the given two boundary conditions are sufficient to solve the system. To solve this system of difference equations, one can use easily by Discrete Invariant Imbedding Algorithm described in the next section.

### 5.2.1.1. Discrete Invariant Imbedding Algorithm

A brief description for solving the tri-diagonal system using Discrete Invariant Imbedding, also called Thomas algorithm is presented as follows:

Consider the scheme:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, 3, \dots, N - 1 \quad (29)$$

Subject to the boundary conditions

$$y_0 = y(0) = \varphi_0; \quad (30)$$

$$y_N = y(1) = \gamma_1. \quad (31)$$

Let us set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \text{ for } i = N - 1, N - 2, \dots, 2, 1. \quad (32)$$

where  $W_i = W(x_i)$  and  $T_i = T(x_i)$  which are to be determined.

From (32) we have:

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \quad (33)$$

By substituting (33) in (29), we have  $E_i (W_{i-1} y_i + T_{i-1}) - F_i y_i + G_i y_{i+1} = H_i$ ,

$y_i = \left( \frac{G_i}{F_i - E_i W_{i-1}} \right) y_{i+1} + \left( \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right)$ , and by comparing with (32), we get the recurrence relations:

$$W_i = \left( \frac{G_i}{F_i - E_i W_{i-1}} \right) \quad (34)$$

$$T_i = \left( \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \quad (35)$$

To solve these recurrence relations for  $i = 1, 2, 3, \dots, N - 1$  we need the initial conditions for  $W_0$  and  $T_0$ . For this we take  $y_0 = \varphi_0 = W_0 y_1 + T_0$ . We choose  $W_0 = 0$  so that the value of  $T_0 = \varphi_0$ . Since  $F_i \geq E_i + G_i$ ;  $E_i > 0, G_i > 0, F_i > 0, W_0 = 0, W_1 < 1 \Rightarrow F_i - E_i W_{i-1} > 0$ ,  $i = 1, 2, \dots, N - 1$ . With these initial values, we compute  $W_i$  and  $T_i$  for  $i = 1, 2, 3, \dots, N - 1$  from (34) and (35) in forward process, and then obtain  $y_i$  in the backward process from (31) and (32) (Sirisha and Reddy, 2014).

### 5.2.2. Right End Boundary Layer Problems

We now discuss the method for singularly perturbed-differential-difference equation of boundary value problems with right-end boundary layer of the underlying interval. Assume that  $p(x)$ ,  $q(x)$  and  $f(x)$  are sufficiently continuously differentiable functions in  $[0, 1]$ . Furthermore, assume that  $p(x) \leq M < 0$  in  $[0, 1]$ , where  $M$  is negative constant. Under these assumptions, (6) has a unique solution  $y(x)$  which exhibits a boundary layer of width  $O(\varepsilon)$  on the right side ( $x = 1$ ) of the underlying interval.

**Lemma 2:** (Doolan *et al.*, 1980 and O'Malley, 1974) Let  $y(x) = y_0 + z_0$  be the zeroth-order asymptotic approximation to the solution of (6), where  $y_0$  represents the zeroth-order approximate outer solution (i.e., the solution of the reduced problem of (6)) and  $z_0$  represents the zeroth-order approximate solution in the boundary layer region of (6).

Then for a fixed positive integer  $i$ ,

$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\gamma(1) - y_0(1)) e^{(p(1) - \frac{\varepsilon q(1)}{p(1)}) (\frac{1}{\varepsilon} - i\rho)} \quad (36)$$

where

$$\rho = \frac{h}{\varepsilon}$$

Proof: let  $y_0(x)$  be the solution of the reduced problem of (6)

$$p(x)y_0'(x) + q(x)y_0(x) = f(x), \quad y_0(0) = \varphi(0) \quad (37).$$

and  $z_0(t)$  is the solution of the boundary value problem of (6) (O'Malley, 1974)

$$z_0''(t) + p(1)z_0'(t) = 0, \quad z_0(1) = \gamma(1) - y_0(1), \quad z_0(\infty) = 0 \quad (38)$$

where  $t = \frac{x}{\varepsilon}$ .

From the theory of singular perturbations, it is known that the solution of (6)-(8) is of the form (O' Malley, 1974)

$$y(x) = y_0(x) + \frac{p(1)}{p(x)} (\gamma(1) - y_0(1)) e^{\int_x^1 \left( \frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)} \right) dx} + o(\varepsilon) \quad (39)$$

As we are considering the differential equations on sufficiently small subintervals or about the point '1', the coefficients could be assumed to be locally constant. Hence,

$$y(x) = y_0(x) + (\gamma(1) - y_0(1)) e^{\int_x^1 \left( \frac{p(1)}{\varepsilon} - \frac{q(1)}{p(1)} \right) dx} + o(\varepsilon) \quad (40)$$

So, at the nodal points, we have

$$y(x_i) = y_0(x_i) + (\gamma(1) - y_0(1)) e^{\left( \frac{p(1)}{\varepsilon} - \frac{q(1)}{p(1)} \right) \left( \frac{1}{\varepsilon} - x_i \right)} + o(\varepsilon), i = 0, 1, \dots, N.$$

that is, 
$$y(ih) = y_0(ih) + (\gamma(1) - y_0(1)) e^{\left( \frac{p(1)}{\varepsilon} - \frac{q(1)}{p(1)} \right) \left( \frac{1}{\varepsilon} - ih \right)} + o(\varepsilon)$$

Therefore  $\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\gamma(1) - y_0(1)) e^{\left( p(1) - \frac{\varepsilon q(1)}{p(1)} \right) \left( \frac{1}{\varepsilon} - i\rho \right)}$  (41)

where  $\rho = \frac{h}{\varepsilon}$

Now introducing a fitting factor  $\sigma$  into equation (23) above we obtain

$$\begin{aligned} \sigma \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) \left( \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right) + \left( p_i + \frac{h^2 p_i}{6\varepsilon} (p_i' + q_i) \right) \left( \frac{y_{i+1} - y_{i-1}}{2h} \right) \\ + \left( q_i + \frac{h^2 p_i q_i'}{6\varepsilon} \right) (y_i) = f_i + \frac{h^2 p_i}{6\varepsilon} f_i' + \tau_i \end{aligned} \quad (42)$$

$y_0 = \varphi_0, y_N = \gamma_1$ . Here  $\sigma$  is a fitting factor which is to be determined in such a way that the solution of (42) with boundary conditions converges uniformly to the solution of (6) – (8) which is in turn a good approximation to the solution of (1) – (3).

Multiplying (42) by  $h$  and taking limit as  $h \rightarrow 0$  we obtain  $\sigma$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sigma}{\rho} \left( 1 + \frac{\rho^2 p^2(ih)}{6} \right) (y(ih + h) - 2y(ih) + y(ih - h)) + \lim_{h \rightarrow 0} \frac{1}{2} p(ih) \left( 1 + \frac{\rho h}{6} \right) (p'(ih) + q(ih)) (y(ih + h) - y(ih - h)) = 0 \end{aligned} \quad (43)$$

where  $\rho = \frac{h}{\varepsilon}$  and  $f_i + \frac{h^2 p_i}{6\varepsilon} f_i' - (q_i + \frac{h^2 p_i q_i'}{6\varepsilon}) y_i$  is bounded.

Now, approximating the solution  $y(x)$  by zeroth-order asymptotic approximation  $y_0(x)$  and using Lemma 2, we have

$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\gamma(1) - y_0(1))e^{\left(p(1) - \frac{\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} \quad (44a)$$

$$\lim_{h \rightarrow 0} y(ih + h) = y_0(0) + (\gamma(1) - y_0(1))e^{\left(p(1) - \frac{\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} e^{-\left(p(1) - \frac{\varepsilon q(1)}{p(1)}\right)\rho} \quad (44b)$$

$$\lim_{h \rightarrow 0} y(ih - h) = y_0(0) + (\gamma(1) - y_0(1))e^{\left(p(1) - \frac{\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon} - i\rho\right)} e^{\left(p(1) - \frac{\varepsilon q(1)}{p(1)}\right)\rho} \quad (44c)$$

By substituting equations (44a) – (44c) in (43), we get the fitting factor as:

$$\sigma = \left(\frac{3\rho p(0)}{6 + \rho^2 p^2(0)}\right) \coth \left[ \left(\frac{p^2(1) - \varepsilon q(1)}{p(1)}\right) \left(\frac{\rho}{2}\right) \right] \quad (45)$$

Finally, making use of equation (42) and equation (45), we get the three term recurrence relation of form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, 3, \dots, N-1 \quad (46)$$

where

$$E_i = \frac{\sigma}{h^2} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) - \frac{p_i}{2h} \left( 1 + \frac{h^2}{6\varepsilon} (p_i' + q_i) \right)$$

$$F_i = \frac{2\sigma}{h^2} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) - \left( q_i + \frac{h^2 p_i q_i'}{6\varepsilon} \right)$$

$$G_i = \frac{\sigma}{h^2} \left( \varepsilon + \frac{h^2 p_i^2}{6\varepsilon} \right) + \frac{p_i}{2h} \left( 1 + \frac{h^2}{6\varepsilon} (p_i' + q_i) \right)$$

$$H_i = f_i + \frac{h^2 p_i f_i'}{6\varepsilon}$$

This gives us the tri-diagonal system which can be solved easily by Discrete Invariant Imbedding Algorithm described in the 5.1.2 above.

### 5.3. Stability of the scheme

We show that the algorithm is computationally stable. By stability, we mean that the effect of an error made in one stage of the calculation is not propagated into larger errors at later stages of the calculations. Let us now examine the recurrence relation given by (34).

Suppose that a small error  $e_{i-1}$  has been made in the calculation of  $W_{i-1}$ ; then, we have

$\bar{W}_{i-1} = W_{i-1} + e_{i-1}$ , where  $\bar{W}_{i-1}$  is the exact value at  $(i-1)$  step and we are actually calculating

$$\bar{W}_i = \left( \frac{G_i}{F_i - E_i \bar{W}_{i-1}} \right) \quad (47)$$

From (47) and (34) we have  $e_i = \bar{W}_i - W_i$  which implies that

$$\begin{aligned} e_i &= \left( \frac{G_i}{F_i - E_i (W_{i-1} + e_{i-1})} \right) - \left( \frac{G_i}{F_i - E_i W_{i-1}} \right) \\ &= \left( \frac{G_i E_i e_{i-1}}{(F_i - E_i (W_{i-1} + e_{i-1})) (F_i - E_i W_{i-1})} \right) \text{ multiplying this equation by } \frac{G_i}{G_i} \text{ we have} \\ &= \left( \frac{W_i^2 E_i}{G_i} \right) e_{i-1} \end{aligned} \quad (48)$$

under the assumption that the error is small initially. From the assumptions made earlier that  $p(x) > 0, q(x) \leq 0$ , we have  $F_i \geq E_i + G_i; |E_i| > 0, |G_i| > 0, |F_i| > 0, i = 1, 2, \dots, N-1$

From (34) we have  $W_1 = \frac{G_1}{F_1} < 1$ , since  $F_1 > G_1$

$$W_2 = \frac{G_2}{F_2 - E_2 W_1} < \frac{G_2}{F_2 - E_2} < \frac{G_2}{E_2 + G_2 - E_2} = 1$$

Then, it follows that  $|e_i| = |W_i|^2 \left| \frac{E_i}{G_i} \right| |e_{i-1}| < |e_{i-1}|$ , since  $|E_i| \leq |G_i|$ .

In the above equation the first inequality is due to  $W_1 < 1$  and the second inequality holds because  $F_2 \geq E_2 + G_2$ . Therefore the recurrence relation (34) is stable (Reddy and Gemmechis, 2012).

Also let us examine the recurrence relation given by (35). Suppose that a small error  $e_{i-1}$  has been made in the calculation of  $T_{i-1}$ ; then, we have

$\bar{T}_{i-1} = T_{i-1} + e_{i-1}$ , where  $\bar{T}_{i-1}$  is the exact value at  $(i-1)$  step and actually calculating

$$\bar{T}_i = \left( \frac{E_i \bar{T}_{i-1} - H_i}{F_i - E_i \bar{W}_{i-1}} \right) \quad (49)$$

From (49) and (35), we have  $e_i = \bar{T}_i - T_i$  which implies that

$$\begin{aligned}
e_i &= \left( \frac{E_i(T_{i-1} + e_{i-1}) - H_i}{F_i - E_i W_{i-1}} \right) - \left( \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right) \\
&= \frac{E_i e_{i-1}}{F_i - E_i W_{i-1}} \text{ multiplying this equation with } \frac{E_i T_{i-1} - H_i}{E_i T_{i-1} - H_i} \text{ we have} \\
&= \left( \frac{T_i E_i}{E_i T_{i-1} - H_i} \right) e_{i-1} \tag{50}
\end{aligned}$$

under the assumption that the error is small initially. From the assumptions made earlier that  $p(x) > 0, q(x) \leq 0$ , we have  $F_i \geq E_i + G_i, W_i < 1, H_i = 0; i = 1, 2, 3, \dots, N - 1$

From (35) we have  $T_1 = \frac{E_1}{F_1} < 1$ , since  $F_1 > E_1$

$$T_2 = \frac{E_2 T_1}{F_2 - E_2 W_1} < \frac{E_2}{F_2 - E_2} < \frac{E_2}{E_2 + G_2 - E_2} = \frac{E_2}{G_2} < 1.$$

In the above equation the first inequality is due to  $T_1 < 1$  and  $W_1 < 1$  also the second and the third inequality holds because  $F_2 \geq E_2 + G_2$  and  $E_2 < G_2$  respectively.

Then it follows that  $|e_i| = |T_i| \left| \frac{E_i}{E_i T_{i-1} - H_i} \right| |e_{i-1}| < |e_{i-1}|$ , since  $|E_i| \leq |E_i T_{i-1}|$ .

Therefore the recurrence relation (35) is stable. Thus the Algorithm is stable for the given finite difference method.

## 5.4. Convergence of the scheme

Writing the tri-diagonal system (28) in matrix-vector form, we get

$$AY = C \tag{51}$$

where,  $A = (m_{ij}), 1 \leq i, j \leq N - 1$  is a tri-diagonal matrix of order  $N - 1$ , with

$$\begin{aligned}
m_{i \ i+1} &= -\sigma\varepsilon - \frac{hp_i}{2} - \frac{h^2\sigma}{6\varepsilon} p_i^2 - \frac{h^3}{12\varepsilon} p_i(p'_i + q_i) \\
m_{i \ i} &= 2\sigma\varepsilon + \frac{h^2\sigma}{3\varepsilon} p_i^2 - h^2 q_i - \frac{h^4 p_i q'_i}{6\varepsilon} \\
m_{i \ i-1} &= -\sigma\varepsilon + \frac{hp_i}{2} - \frac{h^2\sigma}{6\varepsilon} p_i^2 + \frac{h^3}{12\varepsilon} p_i(p'_i + q_i)
\end{aligned}$$

and  $C = (d_i)$  is a column vector with  $d_i = -h^2 f_i - \frac{h^4 p_i f'_i}{6\varepsilon}$ , where  $i = 1, 2, \dots, N - 1$  with local

$$\text{truncation error } T_i(h) = \frac{h^4}{12} K + o(h^6) \tag{52}$$

where  $K = (\varepsilon - \frac{h^2 p_i^2}{\varepsilon}) y_i^{(4)}$  and  $Y = (y_0, y_1, \dots, y_N)^t$ .

$$\text{We also have} \quad A\bar{Y} - T(h) = C \quad (53)$$

where  $\bar{Y} = (\bar{y}_0, \bar{y}_1, \dots, \bar{y}_N)^t$  denotes the actual solution and  $T(h) = (T_1(h), T_2(h), \dots, T_N(h))^t$  is the local truncation error.

$$\text{From equations (51) and (53), we get } A(\bar{Y} - Y) = T(h) \quad (54).$$

$$\text{Thus, we obtain the error equation } AE = T(h) \quad (55),$$

where  $E = \bar{Y} - Y = (e_0, e_1, \dots, e_N)^t$ .

Let  $S_i$  be the sum of elements of the  $i^{\text{th}}$  row of the  $A$ , then we have

$$S_1 = S_i = \sum_{j=1}^{N-1} m_{1,j} = \sigma\varepsilon - \frac{hp_i}{2} + \frac{h^2\sigma}{6\varepsilon} p_i^2 - h^2 q_i - \frac{h^3}{12\varepsilon} p_i(p_i' + q_i) + o(h^4), i = 1$$

$$S_i = \sum_{j=1}^{N-1} m_{i,j} = -h^2 q_i + o(h^4), i = 2, 3, \dots, N - 2$$

$$S_{N-1} = S_i = \sum_{j=1}^{N-1} m_{N-1,j} = \sigma\varepsilon + \frac{hp_i}{2} + \frac{h^2\sigma}{6\varepsilon} p_i^2 - h^2 q_i + \frac{h^3}{12\varepsilon} p_i(p_i' + q_i) + o(h^4), i = N - 1$$

Since  $0 < \varepsilon \ll 1$  and  $\delta = o(\varepsilon)$  for sufficiently small  $h$  also the matrix  $A$  is irreducible and monotone (Mohanty and Jha, 2005) hence an  $n \times n$  matrix  $A$  is said to be irreducible matrix if and only if for some permutation matrix  $P$ , the matrix  $P^t A P$  is not block upper triangular and a real square matrix  $A$  is monotone if for all real vectors  $Y$ ,  $AY \geq 0 \Rightarrow Y \geq 0$ . Then it follows that  $A^{-1}$  exists and elements are non-negative.

$$\text{Hence, from equation (55) we get } E = A^{-1}T(h) \quad (56)$$

$$\text{and } \|E\| \leq \|A^{-1}\| \cdot \|T(h)\| \quad (57)$$

Let  $\bar{m}_{k,i}$  be the  $(k, i)^{\text{th}}$  elements of  $A^{-1}$ . Since  $\bar{m}_{k,i} \geq 0$ , from the theory of matrices we have

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} S_i = 1, k = 1, 2, \dots, N - 1 \quad (58)$$

$$\text{Therefore } \sum_{i=1}^{N-1} \bar{m}_{k,i} \leq \frac{1}{\min_{1 \leq i \leq N-1} S_i} = \frac{1}{h^2 |q_i|} \quad (59)$$

We define  $\|A^{-1}\| = \max_{1 \leq k \leq N-1} \sum_{i=1}^{N-1} |\bar{m}_{k,i}|$  and  $\|T(h)\| = \max_{1 \leq i \leq N-1} |T_i(h)|$ .

From equations (52), (56), (57) and (59), we obtain

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k,i} T_i(h), j = 1, 2, \dots, N-1$$

$$\text{which implies } e_j \leq \frac{h^2 k}{|q_i|}, j = 1, 2, \dots, N-1 \quad (60)$$

where  $k$  is constant. Therefore, we are using the definitions and equation (60) we have

$$\|E\| = o(h^2)$$

Hence, the method gives a quadratic order i.e., the method is convergence for uniform mesh (Sirisha and Reddy, 2014).

## 5.5. Numerical Examples

To demonstrate the applicability of the method, we have applied the method on four boundary value problems. These examples have been chosen because exact solutions are available for comparison. The exact solution of the boundary value problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x - \delta) + c(x)y(x) + d(x)y(x + \eta) = f(x), \forall x \in [0, 1] \quad (61)$$

under the boundary conditions

$$y(x) = \varphi(x), \text{ on } -\delta \leq x \leq 0 \quad (62)$$

$$y(x) = \gamma(x), \text{ on } 1 \leq x \leq 1 + \eta \quad (63)$$

having constant coefficients (i.e.,  $a(x) = a, b(x) = b, c(x) = c, d(x) = d, f(x) = f$ ,

$\varphi(x) = \varphi, \gamma(x) = \gamma$  are constants) is given by:

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \frac{f}{q} \quad (64)$$

where  $q = (b + c + d)$ ,

$$C_1 = \frac{(-f + \gamma q + e^{m_2}(f - \varphi q))}{q(e^{m_1} - e^{m_2})}$$

$$C_2 = \frac{(f - \gamma q + e^{m_1}(-f + \varphi q))}{q(e^{m_1} - e^{m_2})}$$

$$m_1 = \frac{-p + \sqrt{(p^2 - 4q\varepsilon)}}{2\varepsilon}$$

$$m_2 = \frac{-p - \sqrt{(p^2 - 4q\varepsilon)}}{2\varepsilon}$$

### 5.5.1. Impact of parameters $\delta$ and $\eta$ with $\varepsilon$ for the left-end boundary layer

In this section we fix the values of  $\eta$  and  $\varepsilon$  but vary the values of  $\delta$  to investigate its impact on the numerical solution.

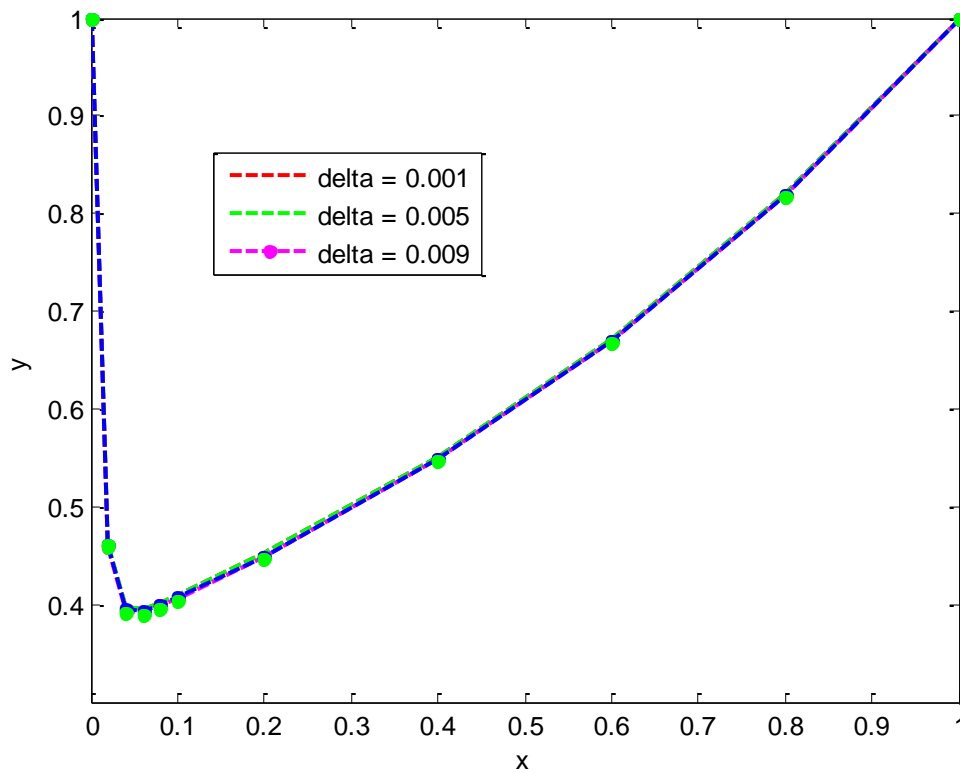
**Example 5.1** (Reddy and Gemmechis, 2012) Consider the model boundary value problem for the left end-boundary layer  $\varepsilon y''(x) + y'(x) + 2y(x - \delta) - 3y(x) = 0, \forall x \in [0,1]$  with  $\varphi = 1, \gamma = 1$ . The exact solution of this problem is given by (64).

The numerical results are given in tables 1, 2 for  $\varepsilon = 0.01$  and  $0.005$  respectively.

**Table 1:** Numerical Results of Example 5.1 for  $\varepsilon = 0.01, N = 100$

	$\delta = 0.001, \eta = 0.005$			$\delta = 0.005 = \eta$			$\delta = 0.009, \eta = 0.005$		
x	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.02	0.4604	0.4620	0.0016	0.4592	0.4608	0.0016	0.4581	0.4597	0.0016
0.04	0.3963	0.3969	0.0006	0.3938	0.3944	0.0006	0.3913	0.3919	0.0006
0.06	0.3946	0.3950	0.0004	0.3918	0.3922	0.0004	0.3889	0.3893	0.0004
0.08	0.4013	0.4016	0.0003	0.3984	0.3987	0.0003	0.3955	0.3958	0.0003
0.10	0.4091	0.4094	0.0003	0.4063	0.4066	0.0003	0.4034	0.4037	0.0003
0.20	0.4518	0.4521	0.0003	0.4490	0.4493	0.0003	0.4461	0.4464	0.0003
0.40	0.5511	0.5514	0.0003	0.5485	0.5488	0.0003	0.5459	0.5461	0.0002
0.60	0.6722	0.6724	0.0002	0.6701	0.6703	0.0002	0.6679	0.6681	0.0002
0.80	0.8199	0.8200	0.0001	0.8186	0.8187	0.0001	0.8173	0.8174	0.0001
1.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000

From table 1 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error decreases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed  $\eta = 0.005$  and varied  $\delta = 0.001, 0.005$  and  $0.009$  and observed that the numerical solution becomes better as the values of  $\delta$  increases with the values of  $\varepsilon = 0.01$  constant and uniform step size.



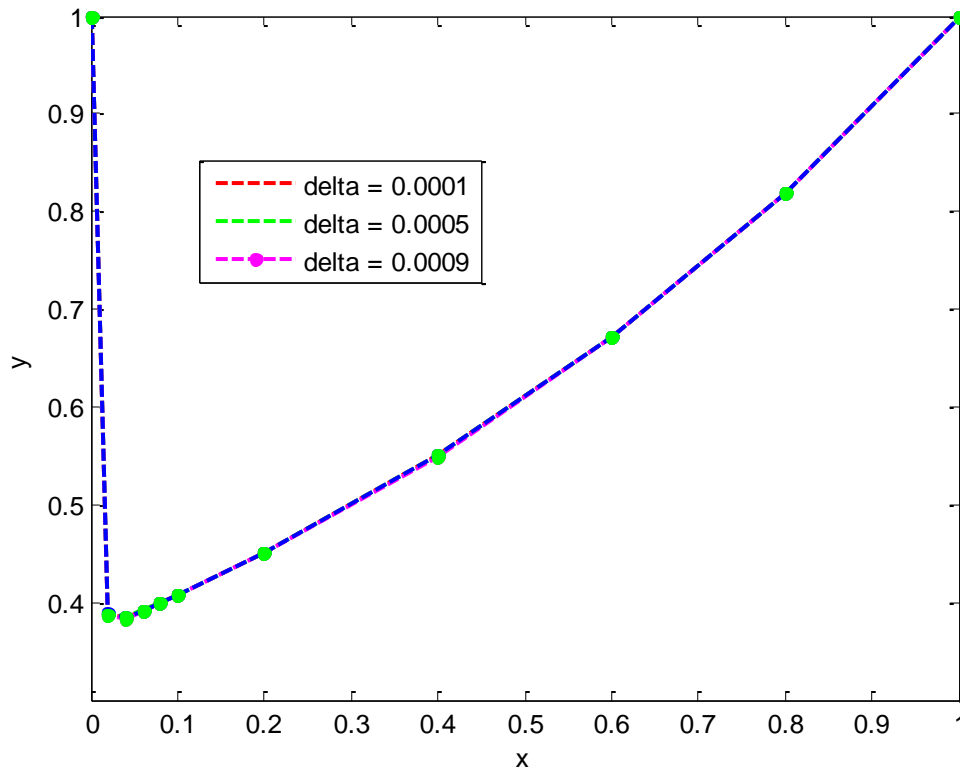
**Figure 1** Graph of Numerical solutions of Example 5.1 of table 1 for  $\varepsilon = 0.01$ ,  $\eta = 0.005$  and different values of  $\delta$

Effect of the parameters of table 1 is shown on figure 1 by considering  $\delta$  increases from 0.001 to 0.009,  $\eta = 0.005$  with  $\varepsilon = 0.01$  for left layer problems. The parameter considered ( $\delta$ ) is does not affect the solution.

**Table 2:** Numerical Results of Example 5.1 for  $\varepsilon = 0.005, N = 100$ 

	$\delta = 0.0001, \eta = 0.0005$			$\delta = 0.0005 = \eta$			$\delta = 0.0009, \eta = 0.0005$		
X	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.02	0.3878	0.3884	0.0006	0.3875	0.3881	0.0006	0.3873	0.3879	0.0006
0.04	0.3842	0.3848	0.0006	0.3839	0.3845	0.0006	0.3836	0.3842	0.0006
0.06	0.3917	0.3923	0.0006	0.3914	0.3920	0.0006	0.3912	0.3918	0.0006
0.08	0.3996	0.4002	0.0006	0.3993	0.3999	0.0006	0.3990	0.3996	0.0006
0.10	0.4077	0.4083	0.0006	0.4074	0.4080	0.0006	0.4071	0.4077	0.0006
0.20	0.4504	0.4510	0.0006	0.4501	0.4507	0.0006	0.4498	0.4504	0.0006
0.40	0.5498	0.5504	0.0006	0.5495	0.5501	0.0006	0.5493	0.5498	0.0005
0.60	0.6711	0.6715	0.0004	0.6709	0.6713	0.0004	0.6707	0.6711	0.0004
0.80	0.8192	0.8195	0.0003	0.8191	0.8193	0.0002	0.8190	0.8192	0.0002
1.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000

From table 2 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error decreases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed small  $\eta = 0.0005$  and varied  $\delta=0.0001, 0.0005$  and  $0.0009$  and observed that the numerical solution becomes better as the values of  $\delta$  is small and increases with the values of  $\varepsilon = 0.005$  constant and uniform step size.



**Figure 2** Graph of Numerical solutions of Example 5.1 of table 2 for  $\varepsilon = 0.005$ ,  $\eta = 0.0005$  and different values of  $\delta$

Effect of the parameters of table 2 is shown on figure 2 by considering  $\delta$  increases from 0.0001 to 0.0009,  $\eta = 0.0005$  with  $\varepsilon = 0.005$  for left layer problems. The parameter considered ( $\delta$ ) is doesn't affect the solution.

### 5.5.2. Impact of parameters $\delta$ and $\eta$ with $\varepsilon$ for the left-end boundary layer

In this section we fix the values of  $\delta$  and  $\varepsilon$  but vary the values of  $\eta$  to investigate its impact on the numerical solution.

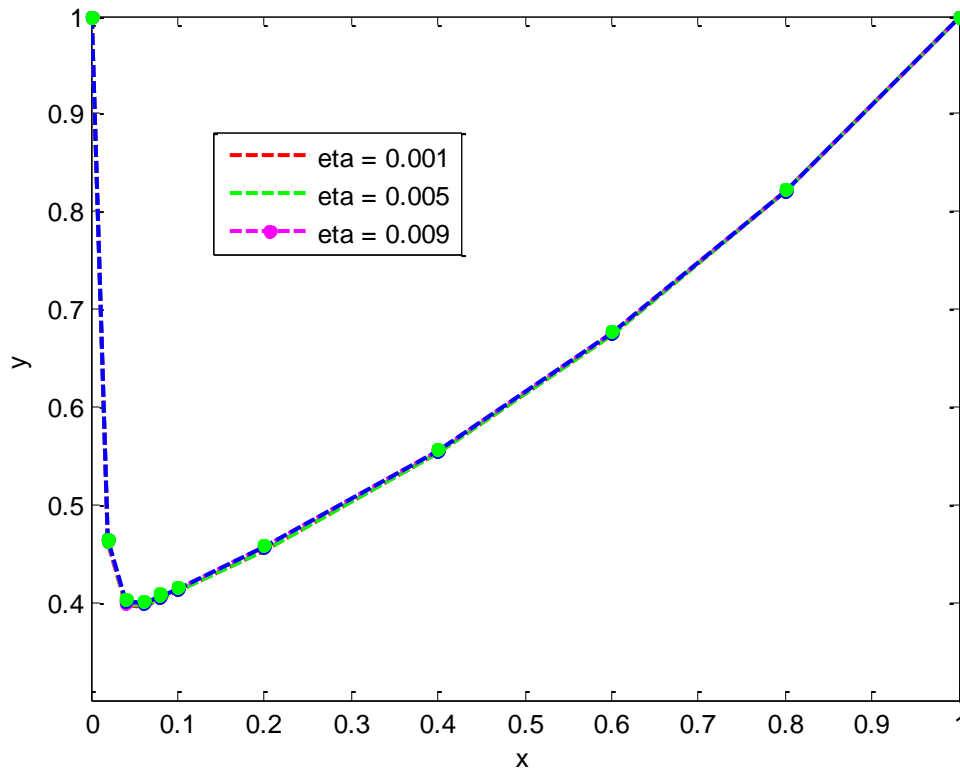
**Example 5.2** (Reddy and Gemmechis, 2012) Consider the model boundary value problem for the left end-boundary layer  $\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x + \eta) = 0, \forall x \in [0,1]$  with  $\varphi = 1, \gamma = 1$ . The exact solution of this problem is given by (64).

The numerical results are given in tables 3, 4 for  $\varepsilon = 0.01$  and  $0.005$  respectively.

**Table 3:** Numerical Results of Example 5.2 for  $\varepsilon = 0.01, N = 100$

	$\delta = 0.005, \eta = 0.001$			$\delta = 0.005 = \eta$			$\delta = 0.005, \eta = 0.009$		
X	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.02	0.4610	0.4626	0.0016	0.4622	0.4638	0.0016	0.4635	0.4650	0.0015
0.04	0.3975	0.3981	0.0006	0.4000	0.4006	0.0006	0.4025	0.4031	0.0006
0.06	0.3960	0.3964	0.0004	0.3988	0.3992	0.0004	0.4016	0.4019	0.0003
0.08	0.4027	0.4030	0.0003	0.4055	0.4058	0.0003	0.4084	0.4087	0.0003
0.10	0.4106	0.4109	0.0003	0.4134	0.4137	0.0003	0.4163	0.4166	0.0003
0.20	0.4532	0.4535	0.0003	0.4560	0.4563	0.0003	0.4588	0.4591	0.0003
0.40	0.5524	0.5527	0.0003	0.5549	0.5552	0.0003	0.5575	0.5577	0.0002
0.60	0.6732	0.6734	0.0002	0.6753	0.6755	0.0002	0.6774	0.6775	0.0001
0.80	0.8205	0.8206	0.0001	0.8218	0.8219	0.0001	0.8230	0.8231	0.0001
1.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000

From table 3 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error decreases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed  $\delta = 0.005$  and varied  $\eta = 0.001, 0.005$  and  $0.009$  and observed that the numerical solution becomes better as the values of  $\eta$  increases with the values of  $\varepsilon = 0.01$  constant and uniform step size.



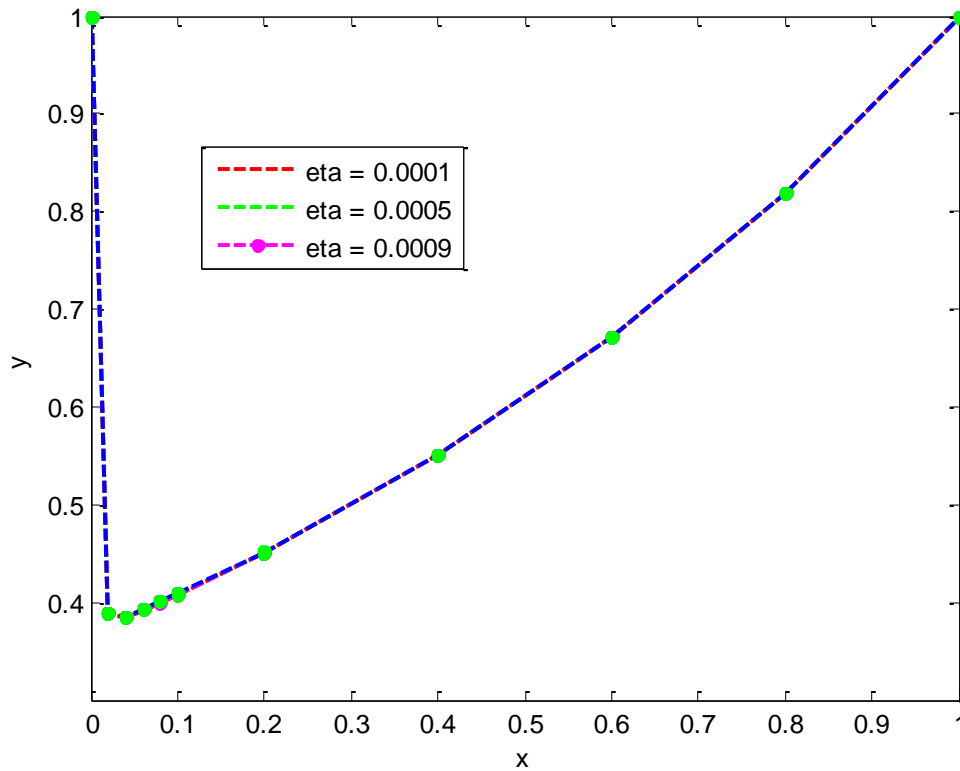
**Figure 3** Graph of Numerical solutions of Example 5.2 of table 3 for  $\varepsilon = 0.01$ ,  $\delta = 0.005$  and different values of  $\eta$

Effect of the parameters of table 3 is shown on figure 3 by considering  $\eta$  increases from 0.001 to 0.009,  $\delta = 0.005$  with  $\varepsilon = 0.01$  for left layer problems. The parameter considered ( $\eta$ ) is not affecting the solution.

**Table 4:** Numerical Results of Example 5.2 for  $\varepsilon = 0.005, N = 100$ 

X	$\delta = 0.0005, \eta = 0.0001$			$\delta = 0.0005 = \eta$			$\delta = 0.0005, \eta = 0.0009$		
	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.02	0.3879	0.3885	0.0006	0.3882	0.3888	0.0006	0.3884	0.3890	0.0006
0.04	0.3843	0.3849	0.0006	0.3846	0.3852	0.0006	0.3849	0.3855	0.0006
0.06	0.3919	0.3925	0.0006	0.3922	0.3928	0.0006	0.3925	0.3931	0.0006
0.08	0.3998	0.4004	0.0006	0.4001	0.4007	0.0006	0.4003	0.4009	0.0006
0.10	0.4078	0.4084	0.0006	0.4081	0.4087	0.0006	0.4084	0.4090	0.0006
0.20	0.4506	0.4511	0.0006	0.4508	0.4514	0.0006	0.4511	0.4517	0.0006
0.40	0.5499	0.5505	0.0006	0.5502	0.5507	0.0005	0.5505	0.5510	0.0005
0.60	0.6712	0.6717	0.0005	0.6714	0.6719	0.0005	0.6717	0.6721	0.0004
0.80	0.8193	0.8196	0.0003	0.8194	0.8197	0.0003	0.8195	0.8198	0.0003
1.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000

From table 4 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error decreases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed small  $\delta = 0.0005$  and varied  $\eta = 0.0001, 0.0005$  and  $0.0009$  and observed that the numerical solution becomes better as the values of  $\eta$  is small and increases with the values of  $\varepsilon = 0.005$  constant and uniform step size.



**Figure 4** Graph of Numerical solutions of Example 5.2 of table 4 for  $\varepsilon = 0.005$ ,  $\delta = 0.0005$  and different values of  $\eta$

Effect of the parameters of table 4 is shown on figure 4 by considering  $\eta$  increases from 0.0001 to 0.0009,  $\delta = 0.0005$  with  $\varepsilon = 0.005$  for left layer problems. The parameter considered ( $\eta$ ) is not affecting the solution.

### 5.5.3. Impact of parameters $\delta$ and $\eta$ with $\varepsilon$ for the right-end boundary layer

In this section we fix the values of  $\eta$  and  $\varepsilon$  but vary the values of  $\delta$  to investigate its impact on the numerical solution.

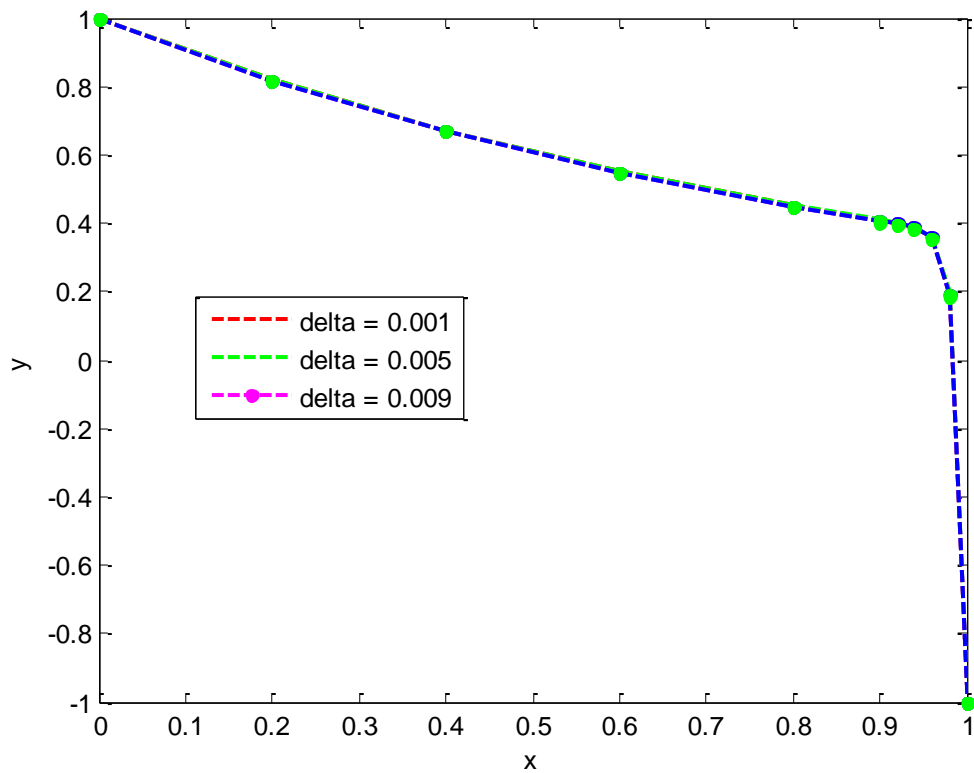
**Example 5.3** (Reddy and Gemmechis, 2012) Consider the model boundary value problem for right end-boundary layer  $\varepsilon y''(x) - y'(x) - 2y(x - \delta) + y(x) = 0, \forall x \in [0,1]$  with  $\varphi = 1, \gamma = -1$ . The exact solution of this problem is given by (64).

The numerical results are given in tables 5, 6 for  $\varepsilon = 0.01$  and 0.005 respectively.

**Table 5:** Numerical Results of Example 5.3 for  $\varepsilon = 0.01, N = 100$

X	$\delta = 0.001, \eta = 0.005$			$\delta = 0.005 = \eta$			$\delta = 0.009, \eta = 0.005$		
	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.20	0.8199	0.8200	0.0001	0.8186	0.8187	0.0001	0.8173	0.8174	0.0001
0.40	0.6722	0.6724	0.0002	0.6701	0.6703	0.0002	0.6679	0.6681	0.0002
0.60	0.5511	0.5514	0.0003	0.5485	0.5488	0.0003	0.5459	0.5462	0.0003
0.80	0.4518	0.4521	0.0003	0.4490	0.4493	0.0003	0.4461	0.4464	0.0003
0.90	0.4091	0.4094	0.0003	0.4062	0.4065	0.0003	0.4033	0.4036	0.0003
0.92	0.4007	0.4010	0.0003	0.3978	0.3981	0.0003	0.3948	0.3951	0.0003
0.94	0.3901	0.3904	0.0003	0.3870	0.3873	0.0003	0.3840	0.3842	0.0002
0.96	0.3619	0.3622	0.0003	0.3583	0.3580	0.0003	0.3547	0.3541	0.0006
0.98	0.1982	0.2009	0.0027	0.1929	0.1917	0.0012	0.1875	0.1847	0.0028
1.00	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000

From table 5 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error increases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed  $\eta = 0.005$  and varied  $\delta = 0.001, 0.005$  and 0.009 and observed that the numerical solution becomes better as the values of  $\delta$  decreases means that not better as the values of  $\delta$  increases with the values of  $\varepsilon = 0.01$  constant and uniform step size.



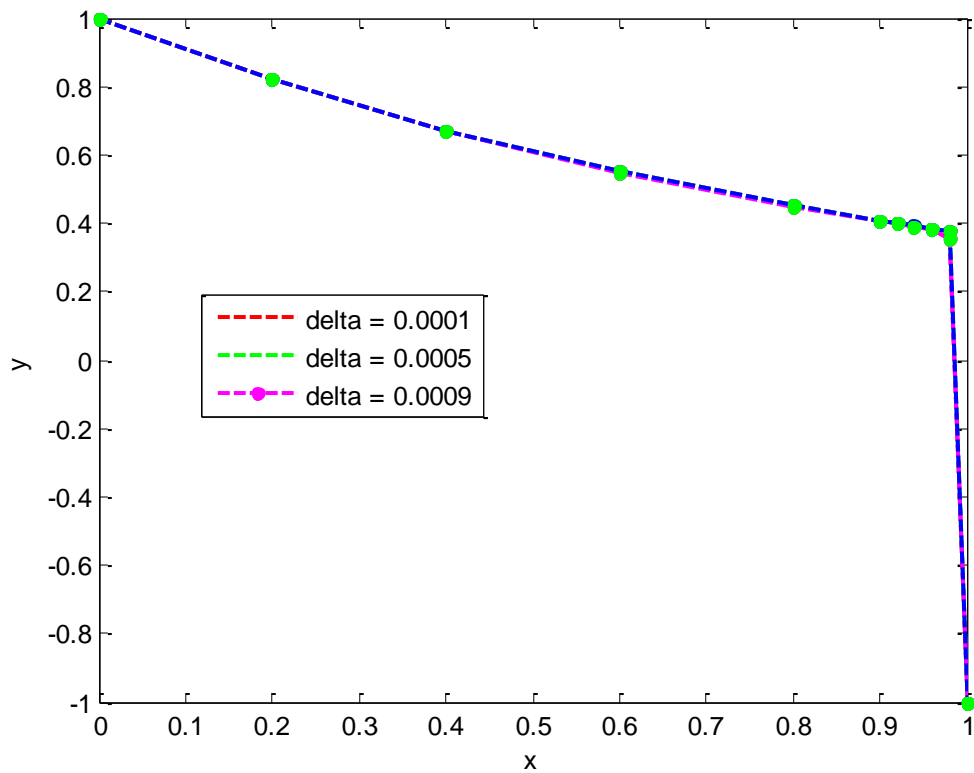
**Figure 5** Graph of Numerical solutions of Example 5.3 of table 5 for  $\varepsilon = 0.01$ ,  $\eta = 0.005$  and different values of  $\delta$

Effect of the parameters of table 5 is shown on figure 5 by considering  $\delta$  increases from 0.001 to 0.009,  $\eta = 0.005$  with  $\varepsilon = 0.01$  for right layer problems. The parameter considered ( $\delta$ ) is not affecting the solution.

**Table 6:** Numerical Results of Example 5.3 for  $\varepsilon = 0.005, N = 100$ 

X	$\delta = 0.0001, \eta = 0.0005$			$\delta = 0.0005 = \eta$			$\delta = 0.0009, \eta = 0.0005$		
	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.20	0.8192	0.8195	0.0003	0.8191	0.8194	0.0003	0.8190	0.8193	0.0003
0.40	0.6711	0.6715	0.0004	0.6709	0.6713	0.0004	0.6707	0.6711	0.0004
0.60	0.5498	0.5503	0.0005	0.5495	0.5501	0.0006	0.5493	0.5499	0.0006
0.80	0.4504	0.4510	0.0006	0.4501	0.4507	0.0006	0.4498	0.4504	0.0006
0.90	0.4077	0.4083	0.0006	0.4074	0.4080	0.0006	0.4071	0.4077	0.0006
0.92	0.3996	0.4002	0.0006	0.3993	0.3999	0.0006	0.3990	0.3996	0.0006
0.94	0.3917	0.3923	0.0006	0.3914	0.3920	0.0006	0.3911	0.3917	0.0006
0.96	0.3836	0.3846	0.0010	0.3833	0.3843	0.0010	0.3830	0.3840	0.0010
0.98	0.3517	0.3770	0.0253	0.3513	0.3767	0.0254	0.3510	0.3764	0.0254
1.00	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000

From table 6 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error increases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed small  $\eta = 0.0005$  and varied  $\delta = 0.0001, 0.0005$  and  $0.0009$  and observed that the numerical solution becomes better as the values of  $\delta$  is small and decreases with the values of  $\varepsilon = 0.005$  constant and uniform step size.



**Figure 6** Graph of Numerical solutions of Example 5.3 of table 6 for  $\varepsilon = 0.005$ ,  $\eta = 0.0005$  and different values of  $\delta$

Effect of the parameters of table 6 is shown on figure 6 by considering  $\delta$  increases from 0.0001 to 0.0009,  $\eta = 0.0005$  with  $\varepsilon = 0.005$  for left layer problems. The parameter considered ( $\delta$ ) is not affecting the solution.

#### 5.5.4. Impact of parameters $\delta$ and $\eta$ with $\varepsilon$ for the right-end boundary layer

In this section we fix the values of  $\delta$  and  $\varepsilon$  but vary the values of  $\eta$  to investigate its impact on the numerical solution.

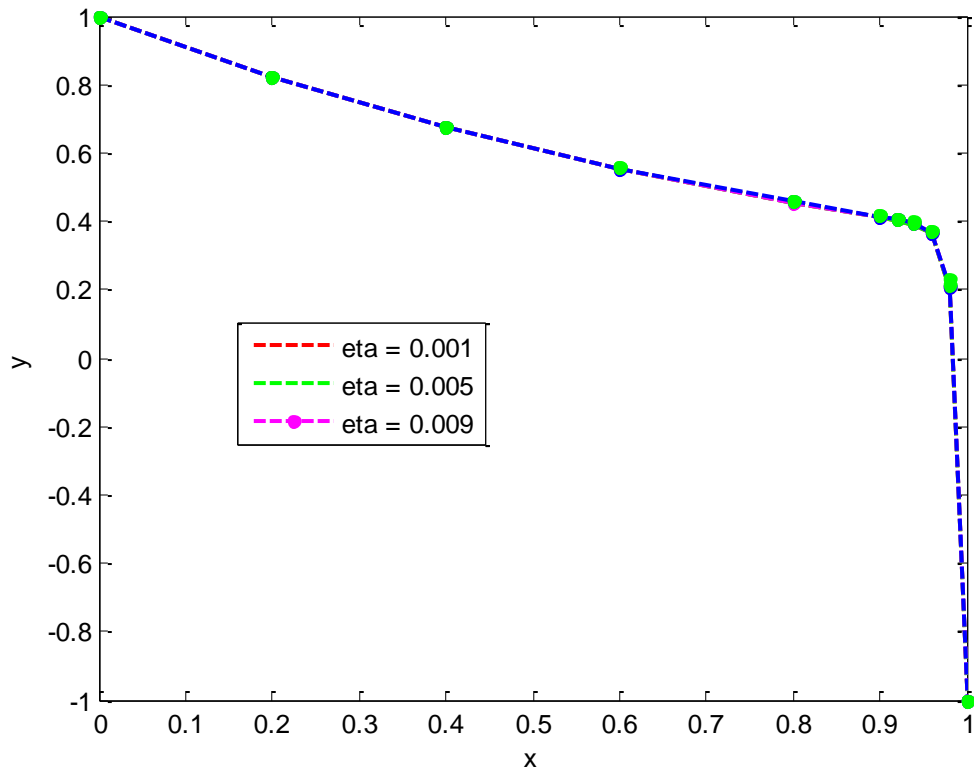
**Example 5.4** (Reddy and Gemmechis, 2012) Consider the model boundary value problem for right end-boundary layer  $\varepsilon y''(x) - y'(x) + y(x) - 2y(x + \eta) = 0, \forall x \in [0,1]$  with  $\varphi = 1, \gamma = -1$ . The exact solution of this problem is given by (64).

The numerical results are given in tables 7, 8 for  $\varepsilon = 0.01$  and  $0.005$  respectively.

**Table 7:** Numerical Results of Example 5.4 for  $\varepsilon = 0.01, N = 100$

X	$\delta = 0.005, \eta = 0.001$			$\delta = 0.005 = \eta$			$\delta = 0.005, \eta = 0.009$		
	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.20	0.8205	0.8206	0.0001	0.8218	0.8219	0.0001	0.8230	0.8231	0.0001
0.40	0.6732	0.6734	0.0002	0.6753	0.6755	0.0002	0.6774	0.6776	0.0002
0.60	0.5524	0.5526	0.0002	0.5549	0.5552	0.0003	0.5575	0.5578	0.0003
0.80	0.4532	0.4535	0.0003	0.4560	0.4563	0.0003	0.4588	0.4591	0.0003
0.90	0.4105	0.4108	0.0003	0.4134	0.4137	0.0003	0.4162	0.4165	0.0003
0.92	0.4021	0.4024	0.0003	0.4050	0.4053	0.0003	0.4079	0.4082	0.0003
0.94	0.3916	0.3918	0.0002	0.3946	0.3948	0.0002	0.3976	0.3981	0.0005
0.96	0.3637	0.3634	0.0003	0.3672	0.3670	0.0002	0.3707	0.3731	0.0024
0.98	0.2009	0.1999	0.0010	0.2062	0.2049	0.0013	0.2114	0.2309	0.0195
1.00	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000

From table 7 one can observe that for fixed values of  $\delta$  and  $\eta$ , the absolute error increases as  $x$  increases (as we move from 0 to 1). In addition, we have experimented with fixed  $\delta = 0.005$  and varied  $\eta = 0.001, 0.005$  and  $0.009$  and observed that the numerical solution becomes better as the values of  $\eta$  decreases with the values of  $\varepsilon = 0.01$  constant and uniform step size.

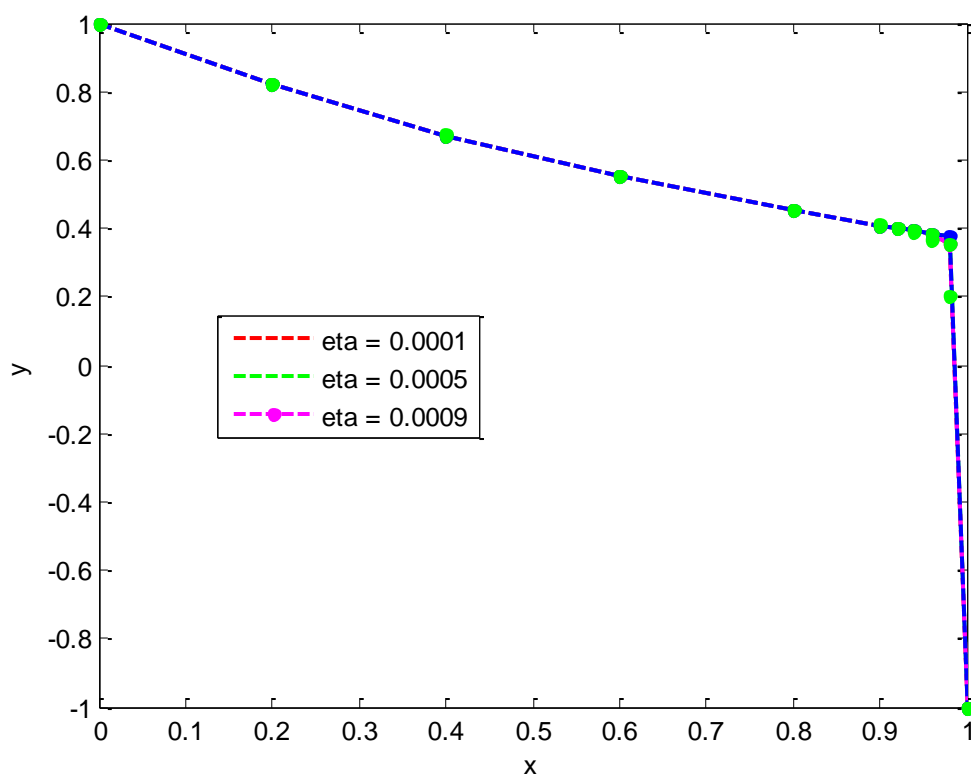


**Figure 7** Graph of Numerical solutions of Example 5.4 of table 7 for  $\varepsilon = 0.01$ ,  $\delta = 0.005$  and different values of  $\eta$

Effect of the parameters of table 7 is shown on figure 7 by considering  $\eta$  increases from 0.001 to 0.009,  $\delta = 0.005$  with  $\varepsilon = 0.01$  for left layer problems. The parameter considered ( $\eta$ ) is not affecting the solution.

**Table 8:** Numerical Results of Example 5.4 for  $\varepsilon = 0.005, N = 100$ 

X	$\delta = 0.0005, \eta = 0.0001$			$\delta = 0.0005 = \eta$			$\delta = 0.0005, \eta = 0.0009$		
	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error	Num. Sol.	Exact Sol.	Absol. Error
0.00	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
0.20	0.8193	0.8195	0.0002	0.8194	0.8197	0.0003	0.8195	0.8206	0.0011
0.40	0.6712	0.6717	0.0005	0.6714	0.6719	0.0005	0.6717	0.6734	0.0017
0.60	0.5499	0.5505	0.0006	0.5502	0.5508	0.0006	0.5505	0.5526	0.0021
0.80	0.4506	0.4511	0.0006	0.4508	0.4514	0.0006	0.4511	0.4534	0.0023
0.90	0.4078	0.4084	0.0006	0.4081	0.4087	0.0006	0.4084	0.4107	0.0023
0.92	0.3998	0.4004	0.0006	0.4001	0.4007	0.0006	0.4003	0.4024	0.0021
0.94	0.3919	0.3925	0.0006	0.3922	0.3928	0.0006	0.3925	0.3918	0.0007
0.96	0.3837	0.3847	0.0010	0.3840	0.3850	0.0010	0.3843	0.3638	0.0205
0.98	0.3519	0.3772	0.0253	0.3523	0.3775	0.0252	0.3526	0.2033	0.1493
1.00	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000	-1.0000	-1.0000	0.0000



**Figure 8** Graph of Numerical solutions of Example 5.4 of table 8 for  $\varepsilon = 0.005$ ,  $\delta = 0.0005$  and different values of  $\eta$

Effect of the parameters of table 8 is shown on figure 8 by considering  $\eta$  increases from 0.0001 to 0.0009,  $\delta = 0.0005$  with  $\varepsilon = 0.005$  for left layer problems. The parameter considered ( $\eta$ ) is not affecting the solution.

Generally, for the left-end layer the absolute errors are given in tables 1, 2, 3 and 4 for different values of the delayed and advanced parameters with different values of perturbations parameter and the effect of small parameters on the boundary layer solution are shown in figures 1, 2, 3 and 4.

Also for the right-end layer the absolute errors are given in tables 5, 6, 7 and 8 for different values of the delayed and advanced parameters with different values of perturbations parameter and the effect of small parameters on the boundary layer solution are shown in figures 5, 6, 7 and 8. But there is no more effect of the parameters when parameters are varying.

Finally, in this study the numerical solution is obtained for different values of delayed and advanced parameters with small values of singular perturbation. But the solution is better for small values of the parameters for left-end boundary layer problems and vice-versa for the right-end boundary layer problems.

### 5.5.5. Order analysis

In this section we will experimentally check whether the order of the method coincides with the theoretical order which is 2. Thus to experimentally check the order of the method is 2 we consider two examples (example 5.1 of table 2 and example 5.2 of table 4) with  $h$  and  $\frac{h}{2}$  are shown as follows

**Table 9:** The order of the method for example 5.1 of table 2

X	h = 0.01			h/2 = 0.005		
	Exact solutions	Num. Sol.	Error ( $e_h$ )	Num. Sol.	Error ( $e_{h/2}$ )	Ratio of $e_{h/2}/e_h$
0.04	0.3848477	0.3842033	0.0006444	0.3846797	0.000168	0.26
0.06	0.3923809	0.3917353	0.0006456	0.3922192	0.0001617	0.25
0.08	0.4002658	0.3996212	0.0006446	0.4001045	0.0001613	0.25
0.10	0.4083128	0.4076695	0.0006433	0.4081519	0.0001609	0.25
0.20	0.4510409	0.4504092	0.0006317	0.4508830	0.0001579	0.249
0.40	0.5503792	0.5498009	0.0005783	0.5502346	0.0001446	0.25
0.60	0.6715958	0.6711253	0.0004705	0.6714782	0.0001176	0.249

This table shows the order of the method for the values parameters  $\delta = 0.0001, \eta = 0.0005$  and  $\varepsilon = 0.005$

**Table 10:** The order of the method for example 5.2 of table 4

X	h = 0.01			h/2 = 0.005		
	Exact Solution	Num. Sol.	Error ( $e_h$ )	Num. Sol.	Error( $e_{h/2}$ )	Ratio of $e_{h/2}/e_h$
0.04	0.3849926	0.3843482	0.0006444	0.3848246	0.000168	0.26
0.06	0.3925262	0.3918806	0.0006456	0.3923646	0.0001616	0.25
0.08	0.4004109	0.3997663	0.0006446	0.4002497	0.0001612	0.25
0.10	0.4084576	0.4078143	0.0006433	0.4082968	0.0001608	0.249
0.20	0.4511832	0.4505515	0.0006317	0.4510252	0.000158	0.25
0.40	0.5505093	0.5499312	0.0005781	0.5503648	0.0001445	0.249
0.60	0.6717017	0.6712313	0.0004704	0.6715841	0.0001176	0.25

This table shows the order of the method for the values parameters  $\delta = 0.0001, \eta = 0.0005$  and  $\varepsilon = 0.005$ .

From table 9 and 10 if the results for step-sizes  $h = 0.01$  and  $h = 0.005$  are compared, it can be seen that the error decreases by a factor of approximately 4 when  $h$  is halved (i.e.,  $e_{h/2}/e_h \approx 1/4$ ). Therefore we can conclude that the order of the method is 2 which coincide with the theoretical order.

## 6. SUMMARY AND CONCLUSION

### 6.1. Summary

A singularly perturbed differential-difference equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay or advance term. The study presented a finite difference method to solve boundary value problems for a second order singularly perturbed differential-difference equations which contains both type of terms having negative shifts as well as positive shifts. First the terms containing the shift parameters was approximated by using Taylor series expansion and the singularly perturbed differential-difference equation was reduced or replaced by an asymptotically equivalent second order singularly perturbed two point boundary value problem. Then the derivation of the reduced second order problem is discretized by second order central difference and a fitting factor was introduced in a finite difference scheme and is obtained from the theory of singular perturbations. Then we obtained a three recurrence relation that can be solved using Thomas algorithm.

The stability and convergence analysis of the method was studied. The solutions exhibits boundary layer behavior at left-end boundary layer or right-end boundary layer depending on the values of the coefficient of the first derivative of the equation reduced problem. The problems and the exact solution are taken from Gemmechis and Reddy in (2012). But the numerical solution was obtained with the MATLAB code of the numerical scheme of this project and solutions have been compared with the exact solutions and absolute errors are presented in tables and effect of small shift on the numerical solution is shown by figures respectively. The numerical results for four test examples demonstrated the efficiency of the method (two examples with left-end boundary layer problems and two examples with right-end boundary layer problems).

## 6.2. Conclusion

In this study, we found the numerical solution of second order singularly perturbed differential-difference equations and for different values of delayed and advanced parameters with small values of singular perturbation. But the solution is better for small values of the parameters for left-end boundary layer problems and vice-versa for the right-end boundary layer problems.

From the results of the study as the values of the parameters increases and perturbations parameter decreases the error decreases for the left-end layer but increases for the right-end layer as  $x$  increases for the constant coefficient problem for those examples. But for small values of parameters the error decreases rather than the other for left and increases for both cases values of the parameters for the right-end boundary layer problems. The method is stable for the left-end layer problem and negligible for the right-end boundary layer problems but convergences for both left-end and right-end boundary layer problems for uniform step size.

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## 8. APPENDIX

MATLAB codes to get the numerical solution of the recurrence relation (28) for left-end boundary layer and right-end boundary layer.

```

function [x,y] = SPDDE1(x0,xf,f,phi,gamma,epsilon,a,b,c,d,delta,eta,h,W0,T0,df,qf,pf)
x=x0:h:xf;
n=length(x);
x=x';
y=zeros(n,1);
y(1)=phi;
x(1)=x0;
x(n)=xf;
W(1)=W0;
T(1)=T0;
y(n)=gamma;
p=(a+(d*eta))-(b*delta);
q=b+c+d;
rho=h/t;
sigma=((3*(rho*p))/(6+(rho^2*p^2)))*coth(((p^2-(epsilon*q))/p)*rho*1/2);
disp( ' x      y ' )
for i=2:n-1
    x(i+1)=x(i)+h;
    E(i)=((sigma/h^2)*(epsilon+((h^2*p^2)/(6*epsilon))))-(p/(2*h))*(1+((h^2/(6*epsilon))*(pf+q)));
    F(i)=(((2*sigma)/h^2)*(epsilon+((h^2*p^2)/(6*epsilon))))-(q+((h^2*p*qf)/(6*epsilon)));
    G(i)=((sigma/h^2)*(epsilon+((h^2*p^2)/(6*epsilon))))+(p/(2*h))*(1+((h^2/(6*epsilon))*(pf+q)));
    H(i)=feval(f,x(i))+((h^2*p*feval(df,x(i)))/(6*epsilon));
    W(i)=G(i)/(F(i)-(E(i)*W(i-1)));
    T(i)=((E(i)*T(i-1))-H(i))/(F(i)-(E(i)*W(i-1)));
end
for i=n-1:-1:1
    y(i)=W(i)*y(i+1)+T(i);
    disp( [ x      y ] )
end
function [x,E] = SPDDEsol(x0,xf,f,phi,gamma,epsilon,a,b,c,d,delta,eta,v)
x=x0:v:xf;
x=x';
n=length(x);
E=zeros(n,1);
y(1)=phi;

```

```

x(1)=x0;
y(n)=gamma;
p=(a+(d*eta)-(b*delta));
q=b+c+d;
M1=(-p+sqrt(p^2-(4*epsilon*q)))/2*epsilon;
M2=(-p-sqrt(p^2-(4*epsilon*q)))/2*epsilon;
C1=((((gamma*q)-feval(f,x)+(exp(M2)*(feval(f,x)-(phi*q)))))/((exp(M1)-exp(M2))*q);
C2=((((feval(f,x)-(gamma*q)+(exp(M1)*((phi*q)-feval(f,x)))))/((exp(M1)-exp(M2))*q);
disp('x E')
for i=2:n-1
    x(i+1)=x(i)+v;
    E(i)=C1*exp(M1*x(i))+C2*exp(M2*x(i))+feval(f,x(i))/q
    disp(['x E'])
end
plot(x,y, 'line specifiers', x,E, 'line specifiers', 'property name', property value); xlabel('x'); ylabel('y');
legend('parameters name=parameters value')

```