A FLUID QUEUE DRIVEN BY SINGLE SERVER MARKOVIAN QUEUE WITH VARIANT WORKING VACATIONS

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I hereby certify that I have read and evaluated this Thesis entitled, **'A FLUID QUEUE DRIVEN BY SINGLE SERVER MARKOVIAN QUEUE WITH VARIANT WORKING VACATIONS'** prepared under my guidance by TSEGAYE LAMORE ADE I recommend that it be submitted as fulfilling the Thesis work requirement.

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DEDICATION

I dedicate this Thesis to my friends for their concern, patience, encouragement and love in the success of my life.

STATEMENT OF THE AUTHOR

By signature below, I declare and affirm that this thesis is my own work and I have followed all ethical and technical principles of scholarship in the preparation and compilation of this Thesis work. Any scholarly matter that is included in the thesis has been given recognition through citation.

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BIOGRAPHICAL SKETCH

The author, Tsegaye Lamore was born in September 1996 GC in South Nations Nationalities and Peoples Region, Hadiya Zone, Mirab Badewacho wereda, Hilfata kebele from his father Mr. Lamore Ade and his mother Mrs. Adenach Wangore. He attended his primary education at kotto Primary school from 2003-2011 Then, he joined Shinshicho Secondary and Preparatory School to attend his secondary education from 2011-2014. He then joined Hawassa University in 2014 and received a Bachelor of Science degree in Mathematics in July 2017. After graduation, he employed at Mirab Badewacho woreda, Wadda secoundary and preparatory school as a mathematics teacher in 2019 and worked for two years. After that, he joined the Postgraduate Program at Haramaya University, College of Natural and Computational Sciences, Department of Mathematics, in September 2021 to pursue a program study leading to a degree of masters of Science in Mathematics with a specialization in Operations Research.

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ABBREVIATIONS

SWV	Single Working Vacation
MWVs	Multiple Working Vacations
VWV	Variant Working Vacation
PGF	Probability Generating Function
QBD	Quasi Birth Death
LT	Laplace Transform
LST	Laplace Steiltjies Transform
SDE	Stochastic Differential Equation
CTMC	Continuous time Markov Chain
FCFS	First come First served
LCFS	Last come First Served
RSS	Random Selection for Service

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A Fluid Queue Driven by Single Server Markovian Queue with Variant Working Vacations

ABSTRACT

In this thesis a fluid queue driven by single server Markovian queue with variant working vacation was investigated. Queuing theory is a mathematical approach that studies and models waiting lines. A fluid Queue is an input output system the customers are modeled as a continuous fluid that enters and leaves a storage device, called Buffer, Where the background process is governed by a single server Markovian queue. Fluid queues are powerfully applied across diverse fields to model systems where a continuous workload accumulates and depletes under rates controlled by a random environment, typically a Markov chain. In telecommunications, they model data buffers in routers and wireless networks with fluctuating traffic and channel capacity. In manufacturing, they represent continuous-flow production lines subject to machine breakdowns, while in finance, they analyze cash reserves in insurance and dam-based company models. The study focuses on formulating governing equations for background process model and fluid model, the closed-form solutions for steady state probabilities of the system were obtained by applying probability generating functions methods. Various performance measures of the system such as buffer content distribution, mean buffer content, server utilization were obtained. The model captures the dynamics of the system under working vacation polices, where the server operates at a reduced rate rather than being completely idle. Numerical computations were performed by MATLAB software to validate the theoretical results and analyze the system performance under various parameter settings. The findings provide insights into the behavior of fluid queues influenced by Markovian driven vacation polices, contributing to the broader understanding of queuing systems with server vacations.

Keywords: *fluid queue, Markovian queue, working vacations, steady-state probabilities, Laplace transform, recurrence relations, probability generating functions.*

1. INTRODUCTION

1.1. Background of the Study

Queuing theory is a mathematical approach that studies the behavior of waiting lines or queues and models the components of a queuing system. A Queuing system can be described as customers arriving for service, waiting for service, if it is not immediate, then leaving the system after being served (Gross *et al*., 2008). In queueing theory, a customer is any entity that arrives at a service point to request a service, whether it's a person, a data packet, or a physical object. The customer's experience of waiting and service is the central metric for evaluating a queueing system**.** The theory models the entire customer journey including arrival, waiting time, service time, and departure to analyze and optimize system performance (Hillier and Lieberman, 2015).

In the classical queuing theory, it is assumed that the server is always available to provide service for customers (Shortle *et al.*, 2018). However, a more realistic situation which is observed in some queuing systems is that the server may become unavailable for a period of time due to various reasons such as utilization of server's idle time, system maintenance, server breakdown, etc. A vacation in a queuing context is a period when the server is not available for providing service. Arrivals coming during the vacation can go into service only after the server returns from the vacation period. The concept of server's vacation in queuing model was introduced by Levy and Yechiali (1975) while analyzing utilization of idle time in an M/G/1 queuing model. They considered a model in which the idle time of the server is utilized for additional work in a secondary service system.

There are two types of WV models known as multiple working vacations (MWVs) and single working vacation (SWV). In the SWV model, when the system is empty, the server goes for vacation and when the vacation ends, if there are customers in the system, the server switches back to its regular service rate; otherwise, it will stay idle until a new customer arrives. But for MWVs at the end of a vacation if there are customers in the system, the server switches back to its regular service rate; otherwise, the server takes another WV and continues this process until it finds at least one waiting customer at the vacation termination epoch. Servi and Finn (2002) introduced working vacations (WVs) for the first time and analyzed a single-server Markovian queue with MWV policy.

In the past two decades queuing models with WV have been well investigated, due to their applications in many areas such as computer systems, communication networks and manufacturing systems. Servi and Finn (2002) analyzed M/M/1 queue with MWV using the probability generating function (PGF). They derived the PGF of the number of customers in the system and sojourn time in steady state and applied their results to performance analysis of gateway router in fiber communication networks. This result was viewed as multi-queue generalization of a cyclic service queuing system in which each queue can be modeled as an M/M/1 queue with WV.

Since the work of these authors, queuing system with WV has attracted a lot of researchers, and several authors have extended the original model. Liu *et al.* (2007) analyzed the same M/M/1/ queue with MWV model using the matrix-geometric method. They obtained explicit expressions of the performance measures. Tian *et al.* (2008) studied the M/M/1 queue with SWV by using quasi birth and death process and matrix-geometric solution method, they obtained busy period and expected busy cycle.

Laxmi and Rajesh (2017) analyzed M/M/1 queue with variant WVs, balking and reneging. They derived PGFs of the steady state probabilities and obtained the closed form expressions of the system size. Laxmi and Kassahun (2020) analyzed M/M/c Queue with variant working vacation under multi server environment. Mao et al. (2012) investigated a fluid model driven by M/M/1 queue with multiple vacations and N-policy. They introduced a new method for solving steady state probabilities of fluid model modified matrix-geometric solution method. Fluid model is suitable for modelling traffic network where individual arrivals has less impact on the performance of the network (Deepa and Mary, 2020)

1.2. Statement of the problem

A fluid Queue is an input output system the customers are modeled as a continuous fluid that enters and leaves a storage device, called Buffer where according to rates that depend on underlying stochastic process that is related to the state of the machine (on, off, under repair or preventive maintenance etc.). Such fluid queues are determined as mathematical tools for high-speed communication networks, transportation systems and production-inventory systems. The common characteristic of these application areas is that the customer-units are processed very fast in comparison with changes in the server status. Indeed, the on-off alterations of the independent sources that occur in fluid queues can be seen as vacations/failures of the server, the gate-keeper or the administrator of the system. Queuing systems with server vacation have been investigated extensively due to their wide applications in several areas including computer system, communication systems, service system, manufacturing and production systems (Yu *et al.,* 2017).

In Working Vacation (WV) models, a fluid queuing model driven by an $M/M/1/N$ queue subject to working vacation studied by Vijayashree and Anjuka (2017) computed buffer content distribution by using Bessel function of the first kind. They successfully derived the joint steady state probability distribution of the system by solving the associated system of differential equations and applying probability generating function.

Darwiesh et al. (2021) considered a fluid queue with an infinite buffer capacity which is both filled and depleted by a fluid at a constant rate. They obtained a simple series form for joint stationary distribution of the buffer occupancy. The method is explicit where the coefficient of the series is obtained in closed form. Vijayashree and Anjuka (2018) studied a fluid model driven by Markovian queue subject to working vacation. The underlying system of differential difference equation that governs the process is solved using continued fraction and generating function methodologies. Many authors studied a fluid queue model driven by working vacations and they used different solution methodologies. But to the best of our knowledge a fluid queue driven by Markovian queue with variant working vacation has not been studied so far. This study investigated a fluid queue driven by single server Markovian queue with variant working vacations to fill the gap.

1.3. Objectives of the study

The main objective of this study is to develop a fluid queue model driven by M/M/1 queue with variant working vacations

The study explored the following specific objectives

- ✓ To formulate the governing equations of the developed model
- ✓ To obtain steady state probabilities of the model
- ✓ To obtain buffer content distribution
- ✓ To Analyze performance measures of the model.

2. LITERATURE REVIEW

2.1. Queuing Models with Working Vacations

Queueing models subject to vacation have attracted the attention of many researchers owing to its widespread applicability in real time situations. The concept of queueing systems with server vacations was first discussed by Levy and Yechiali (1975). Tian and Zhang (2003) analyzed a multi-server (M/M/c) queue with a single vacation policy for some idle servers. They obtained stationary distribution of the system using matrix method analysis. Sikdar and Gupta (2008) considered an M/G/1/N queue with multiple and single vacation policies

Katayama (2011) has investigated the M/G/1 queue with multiple and single vacation, sojourn time limits and balking behavior. Using the level crossing approach, explicit solutions for the stationary virtual waiting time distribution are derived under various assumptions on the service time distribution. Sakuma and Inoie (2012) analyzed the M/M/c queue with multiple vacation exponentially distributed, where customers are impatient only when all servers are unavailable. They derived the stationary distribution of the system using the matrix-analytic method.

Yue *et al.* (2014) considered a variant of the M/M/1 model introduced by Altman and Yechiali (2006), they assumed that the server is allowed to take a maximum number K of vacations if the system remains empty after the end of a vacation. They have obtained closed-form expressions for important performance measures. Yue *et al.* (2014) analyzed M/M/c queue with impatient customers and synchronous vacations. They developed balance equations for the steady state probabilities and solved the equations using PGF method.

Yue *et al.* (2016) extended the model studied by Altman and Yechiali (2006). The authors assumed that customers are impatient regardless of the state of the server. Furthermore, customers have a deadline depending on whether the server is on vacation or busy. They made comparisons between the mean system sizes under the variant vacation policy and the mean system sizes under the single vacation policy or the multiple vacation policy. Servi and Finn (2002) introduced new vacation model called working vacation model, in which during vacation period the server serves with slower rate than that of the normal service period. They analyzed the M/M/1 queue with MWVs to model a Wavelength Division Multiplexing (WDM) optical access network. They derived the PGF of the number of customers in the system and sojourn time in steady state. Since the work of Servi and Finn, queuing system with WV has attracted a lot of researchers, and several authors have extended the original model. Liu *et al.* (2007) analyzed the same M/M/1/MWV model using the matrix-geometric method. They obtained explicit expressions of the performance measures. Tian *et al.* (2008) studied the M/M/1 queue with SWV by using quasi birth and death process and matrix-geometric solution method.

Laxmi and Rajesh (2017) analyzed M/M/1 queue with variant WVs, balking and reneging. They derived PGFs of the steady state probabilities and obtained the closed-form expressions of the system size. The model successfully quantifies how the slower service rate during vacations and the customer impatience rate directly and significantly degrade system performance. This leads to longer queues, higher abandonment rates, and decreased server utilization in its busy state.

Manoharan and Majid (2017) considered an M/M/c queue with multiple working vacation and impatient customers. They obtained the PGF for the number of customers present when the server is both in a service period as well as in a WV period. They have derived a closed-form solution for various performance measures such as the mean queue length and the mean waiting time. Xu et al. (2017) analyzed multi server M/M/c queue with working vacations and they derived steady state distribution of the queue length by using QBD process and matrix-geometric solution method.

2.2. Fluid Queue Models with Vacations

Barbot and Sericola (2002) obtained analytic expression for the joint stationary distribution of the buffer level and the state of the M/M/1 queue by using generating function method. Whitt (2006) investigated Deterministic fluid models were developed to provide simple first order performance descriptions for multi-server queues with abandonments under heavy load.

Mao et al. (2012) introduced a new method of fluid model modified matrix geometric solution method. They derived concise expression for performance measure-mean buffer content. Arunachalam and Dharmaraja (2014) founded steady state distribution of the buffer content of a fluid queue modulated by two independent birth and death process by using differential equation techniques to solve system of equations. They determined inflow rates by birth death process with finite state space and out flow rate from the buffer by current state of another independent birth death with four states evolving in the background.

Vijayashree and Anjuka (2016) analyzed a fluid model driven by M/M/1 Queueing model subject to Bernoulli-schedule controlled vacation and vacation interruption the governing system of differential difference equation is solved using matrix geometric method in the Laplacian domain under consideration of quasi birth death process.

Yu et al. (2017) analyzed a fluid queue driven by multi server queue with multiple working vacations and vacation interruptions. They presented a system of differential equation that satisfy a fluid queue and obtained matrix-geometric structure for Laplace transform of the stationary buffer content. In the same year Vijayashree and Anjuka (2017) studied a fluid queueing model driven by M/M/1/N queue subject to working vacation. The underlying system of differential difference equation that governs the process was solved using Laplace transform and generating function methodology explicit expressions for the joint steady state probabilities of the background queuing model and the content of the buffer are obtained in terms of modified Bessel function of the first kind.

Latouche and Nguyen (2018) focused on telecommunication system modelling and determined stationary buffer content distribution they gave high level presentation of matrix analytic approach to the analysis of fluid models. Xu et al. (2018) investigated a fluid model driven by M/M/1 with working vacation and removal of customer end of negative customer and established a fluid model based on stochastic process and obtained mean buffer content and the probability of empty buffer for this fluid queue using LT (Laplace transform) method.

Paoumy and Radwan (2020) considered a fluid queue driven by truncated queue with discouraged arrivals and acquired steady state distribution of buffer content and stationary state probabilities of background birth death process by using efficient matrix method. Kumar et al (2023) studied fluid flow queue regulated by an M/M/1 queueing system in the presence of active catastrophic failures and subsequent repairs of server. stationary buffer content is determined in terms of modified Bessel function of first kind and derived explicit closed form analytical expression for probability density function.

As we discussed above, different fluid queuing models with working vacations were studied by many researchers. However, there is a limited study in consideration of variants of working vacations. there are many real-life applications that involve fluid models. Therefore, this study focused on fluid queue model with variant working vacations.

3. MATERIALS AND METHODS

This chapter describes the materials and methods that were used to achieve the objectives of the study. The study commenced by establishing important mathematical preliminaries, including definitions, properties, and distributions of various functions, which served as foundational tools throughout the analysis. This foundational work was developed through a comprehensive review of key textbooks, peer-reviewed journal articles, and authoritative online resources. The underlying queueing process was modeled as a continuous-time Markov chain (CTMC), where states represent different vacation and service phases, incorporating variant working vacation policies that allow the server to operate at a reduced rate during vacation periods. Using the Markov property, the Forward Kolmogorov difference equations were derived to formulate the governing equations of the system, accounting for the exponential nature of state transitions in continuous time. The steady-state probabilities of the background queueing model were obtained by solving the resulting balance equations. Through the application of probability generating functions (PGFs), solved under appropriate boundary conditions and normalization constraints, closed-form expressions for these steady-state probabilities were derived.

The fluid queue was then characterized by a stochastic differential equation (SDE) in which the drift rate depends on the state of the background Markov process. To analyze the fluid content distribution, the Laplace-Stieltjes transform (LST) was applied, converting the differential equations into more tractable algebraic forms. The solution of these equations involved modified Bessel functions of the first kind, arising from the structure of the governing differential equations. Recurrence relations between fluid level probabilities were established by examining the coupled system of equations, enabling a detailed description of the time-dependent and stationary behavior of the fluid process.

To ensure robustness, computational simulations were conducted to validate the theoretical results, demonstrating consistency between analytical and empirical outcomes. Furthermore, numerical inversion of Laplace transforms was employed to cross-verify the analytical expressions, reinforcing the validity of the derived solutions. Finally, graphical and tabular outputs generated via MATLAB were used to illustrate the influence of key system parameters on performance measures, providing intuitive insights into the system's behavior under the influence of a Markovian queue with variant working vacations. This structured and multi-faceted approach ensured a rigorous derivation and validation of the model's dynamics.

4. PRELIMINARIES

This chapter provides an overview of key concepts in queueing systems, stochastic processes, and important definitions and properties of special functions that are used throughout this study.

4.1. The Concept of Queueing System

A queuing system can be described as any system whereby a customer arrives for some services, waits for service, receives service and later leaves the system. Some customers may leave without receiving service, perhaps because they grow tired of waiting in line or perhaps because there is no room to enter the service facility in the first place (Shortle *et al*., 2018).

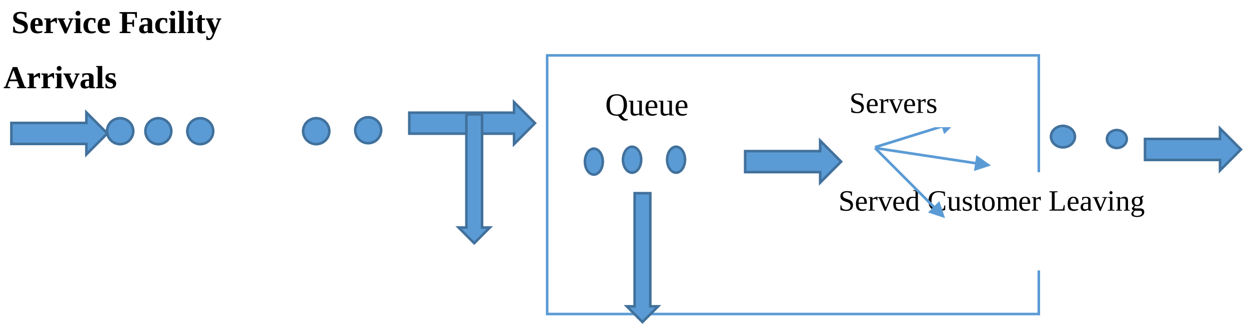

Figure 1 A Typical Queuing System

A quantitative evaluation of a queueing system requires a mathematical characterization of the underlying processes. A queuing system is mainly described by the following four basic characteristics (Bhat ,2008).

The Input Process. It expresses the mode of arrival of customers at the service facilities governed by some probability law. In most cases, the arrivals are the product of external factors. Therefore, the best one can do is to describe the input process in terms of random variables that represent either the number arriving during a time interval or the time interval between successive arrivals.

The Service Mechanism: This means the arrangement of servers facility to serve the customers. The uncertainties involved in the service mechanism are the number of servers, the number of customers getting served at any time, and the duration and mode of service.

System Capacity: The number of customers that can wait at a time in a queueing system is a significant factor for consideration. If the waiting room is large, one can assume that for all practical purposes, it is infinite. In some systems, there is a physical limitation to the amount of space for customers to wait, so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available. These are referred to as finite queueing situations.

Queue discipline: Queue discipline refers to the manner in which customers are selected for service when a queue has formed. A common discipline in everyday life is FCFS. However, there are many other disciplines. Some other queue disciplines are: LCFS and RSS in which customers are selected randomly from the queue independent of their arrival times.

For describing queueing processes, a notation has evolved, due for the most part to Kendall (1953), which is now rather standard throughout the queueing literature. A queueing process is described by a series of symbols and slashes $\mathrm{A/B/X/Y/Z}$, where A denotes the inte-rarrival time distribution, B denotes the service-time distribution, X denotes the number of parallel servers, Y denotes the system capacity, and Z denotes the queue discipline. For example, $\mathrm{M/D/3/INF/FCFS}$ indicates a queueing system with exponential inter-arrival times, deterministic service times, three parallel servers, infinite system and FCFS queue discipline. In many situations only the first three symbols are used. Typical practice is to omit the service capacity if no restriction is imposed $Y = \infty$ and to omit the queue discipline if it is first come, first served (Z = FCFS). Thus $\mathrm{M/D/2}$ would be the same as $\mathrm{M/D/2/\infty/FCFS}$ (Gross *et al.*, 2017).

4.2. Basic Concepts on Stochastic Processes

Uncertainties in model characteristics lead us to random variables as the basic building blocks for the queueing model. We have the following definitions according to Bhat (2015).

Definition 4.1: A random variable is a function that associates a number with each point in an experiment's sample space.

Definition 4.2: A random variable X is **discrete** if it can assume only discrete values $\mathrm{x}_1, \mathrm{x}_2, \ldots$. and a random variable X is **continuous** if, for some interval, the random variable X can assume all values on the interval.

Definition 4.3: Let T be the range of time of interest, Time can be continuous or discrete. We denote the time $t \in T$ when it is continuous, and $n \in T$ when it is discrete. Then the family of random variables $X(t), t \in T$ or the sequence of random variables $X_n, n \in T$ is known as a stochastic process. The space in which $X(\mathrm{t})$ or $\mathrm{X_n}$ assumes values are known as the state space and T is known as the parameter space(index set).

Definition 4.4: A counting process is a stochastic process in which $X(t)$ takes on nonnegative integer values and is non-decreasing in time. A counting process typically represents the cumulative number of events that have occurred by time t.

4.2.1. The exponential distribution.

In queueing theory, the exponential distribution is often used to model the time until a particular event occurs for example, the time until the next arrival or the time until a customer completes service. According to Gross *et al*. (2017), we have the following definitions.

Definition 4.5: An exponential random variable is a continuous random variable T with probability density function (PDF) $f(t)$

$f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ where $\lambda > 0$ is constant

The cumulative distribution function, complementary cumulative distribution function, mean, and variance of an exponential random variable T can be obtained from the PDF

$\boldsymbol{F(t) = Pr\{T \leq t\} = 1 - e^{-\lambda t}}$ **t≥ 0**

$\boldsymbol{G(t) = Pr\{T > t\} = e^{-\lambda t}}$ **t ≥ 0**

$\boldsymbol{E(t) = \frac{1}{\lambda}, Var(t) = \frac{1}{\lambda^2}}$

A key property of the exponential distribution is the memory less property, which is defined as follows

Definition 4.6: A random variable T is said to be without memory or memoryless, if for all $s \geq 0$,and all $t \geq 0$ the following condition holds

$$Pr\{T > s + t \mid T > t\} = \Pr\{T > s\} \ (s, t \ \geq 0)$$

In words, we can think of T as representing the time until some event occurs, for example, the time until the next bus arrives. The memory less property states that if one has already been waiting s time units for a bus to arrive $(T > s)$, then the conditional probability of waiting at least another t units $\{T > s + t \mid T > t\}$ is the same as the probability of waiting at least t units in the first place $Pr\{T > s\}$. The fact that one has been waiting for a certain period of time does not mean that the bus is due to arrive. Rather, a memory-less process continually starts over. The remaining time until the event occurs does not depend on the amount of time spent waiting so far.

4.2.2. The Poisson Process.

The Poisson process is a common process for modeling arrivals to a queueing system. At the same time, the process can be thought of describing events that occur randomly in time. We have definitions 4.7 and 4.8 (Gross *et al.* ,2018).

Definition 4.7: A Poisson process with rate $\lambda > 0$ is a counting process $N(t)$ with the following properties:

$$\mathrm{N}(0) = 0$$

$\Pr\{1 \text{ event between } \mathrm{t} \text{ and } \mathrm{t} + \Delta\mathrm{t}\} = \lambda\Delta\mathrm{t} + \mathrm{o}(\Delta\mathrm{t})$

$\Pr\{2 \text{ or more events between } \mathrm{t} \text{ and } \mathrm{t} + \Delta\mathrm{t}\} = \mathrm{o}(\Delta\mathrm{t})$

The numbers of events in non-overlapping intervals are statistically independent where $o(\Delta t)$ denotes a quantity that becomes negligible when compared to Δt.

Definition 4.8: A Poisson random variable is a discrete random variable X with probability mass function

$P(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}$, $n = 0,1,2,..$ where $\lambda > 0$ is constant

The mean and variance of a Poisson random variable.X are $\mathrm{E}[\mathrm{X}] = \lambda$ and $\mathrm{Var}[\mathrm{X}] = \lambda$.

Continuous time Markov chain

According to Yin and Zhang (2012) we have the following definition for continuous time Markov chain.

Definition 4.9: A continuous time Markov chain is a stochastic process $\{X(t), t \geq 0\}$ with a countable state space, such that:

i) Each time the process enters state $i,$ it remains in that state for a period of time that is exponentially distributed with rate v_i independent of the past

ii) When the process departs state $i,$ it goes to other state j with transition probability P_{ij} independent of the past

In other words, a continuous time Markov chain for each transition from state to state the time spent in each state is an exponential random variable in continuous time; that is a continuous time Markov chain is a stochastic process that moves from state to state such that the amount of time it spent in each state before proceeding to the next state, is exponentially distributed. In addition, the amount of time that the process spent in state i and the next visited state are independent of each other.

In continuous time according to the Markov property can be stated as follows,

$\Pr\{X(t+s)=j \mid X(t)=i, X(u), 0 \leq u < t\} = \Pr\{X(t+s)=j \mid X(t)=i\}$.

The probability of being in state j at future time $t+s$ given only that we're in state i now

Definition 4.10: A finite simple continued fraction is an expression of the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\dots+\cfrac{1}{a_n}}}}}$$

Where a_0 is an integer and $\mathrm{a}_1, \mathrm{a}_2, \dots \mathrm{a}_\mathrm{n}$ is a positive integer a_i are called the partial quotients. For conciseness, this fraction is often denoted by $[\mathrm{a}_0\,; \mathrm{a}_1, \mathrm{a}_2, \dots \mathrm{a}_\mathrm{n}]$ (Karpenkov, 2013)

An infinite simple continued fraction is defined by an infinite sequence of integers $\mathrm{a}_0\,, \mathrm{a}_1, \mathrm{a}_2, \dots$ with $\mathrm{a}_\mathrm{i} > 0$ for $\mathrm{i} \geq 1$ It is denoted by $[\mathrm{a}_0\,; \mathrm{a}_1, \mathrm{a}_2, \dots]$ Its value is defined as the limit of its sequence of convergent. For a continued fraction (finite or infinite) $[\mathrm{a}_0\,; \mathrm{a}_1, \mathrm{a}_2, \dots \mathrm{a}_\mathrm{n}]$ the k^{th} convergent C_k the finite continued fraction truncated at a_k, $C_k = [\mathrm{a}_0\,; \mathrm{a}_1, \mathrm{a}_2, \dots \mathrm{a}_\mathrm{k}] = \frac{P_k}{q_k}$ where p_k *and* $q_k$and are integers chosen to be coprime. The fractions C_k are the fundamental rational approximations to the full continued fraction.

Definition 4.11: A recurrence relation is a mathematical equation that defines a sequence by expressing each term as a function of its preceding terms. This recursive formulation provides a powerful mechanism for modeling discrete processes across computer science, economics, and engineering, where systems evolve based on their prior states (Rosen, 2011). Unlike a closed-form solution, a recurrence inherently captures sequential dependency, making it indispensable for analyzing iterative algorithms and time-discrete dynamical systems (Sedgewick, 2013). Meticulously , for a sequence $\{a_n\}$ a recurrence relation of order k is expressed as:

$$a_n = f(a_{n-1}, a_{n-2}\ , \dots . . a_{n-k}, n)$$

for $n \geq k$The initial conditions $a_0, a_1, \dots a_{k-1}$ are essential to ground the sequence and ensure a unique solution (Levin, 2025).

Definition 4.12: A recurrence is **linear** if its defining function f is a linear combination of its previous terms. The general form is:

$$a_n + c_1(n)a_{n-1} + c_2(n)a_{n-2} + \cdots + c_k a_{n-k} = g(n) \tag{4.1}$$

If all coefficients $c_i(n)$ are constants, it is a linear recurrence with constant coefficients, which admits well established analytical solutions

A recurrence that cannot be expressed in this additive form is nonlinear, often requiring specialized techniques like linearization or computational methods.

Definition 4.13: The linear recurrence is homogeneous if the forcing term $g(n) = 0$ for all n

$$a_n + c_1 a_{n-1} + \cdots . + c_k a_{n-k} = 0 \tag{4.2}$$

If $g(n) \neq 0$ for the equation (4.1), then the linear recurrence is non-homogeneous. The solution is constructed as the sum of the general solution to the homogeneous equation and a particular solution to the non-homogeneous one (Levin, 2025).

4.3. Essential Principles of Fluid Queuing Systems

Fluid Queuing Systems Overview

Fluid queuing models are continuous approximations of conventional discrete queuing systems, that treat jobs (or customers) as a continuous flow vs. separate entities. Fluid queuing models are well suited for evaluating systems with significant traffic where discrete-event simulations become computationally intensive.

Continuous flow: work is conceived as a fluid (similar to water moving through a pipe) rather than separable units.

Deterministic or stochastic: Fluid can be considered as deterministic (a constant flow rate) or stochastic (fluid has random variation).

Scalability: Ideal for large systems, such as telecommunications networks, traffic flow, and a manufacturer’s production process.

4.4. Modified Bessel function of first kind and Laplace transform

4.4.1. Modified Bessel function of first kind

The modified Bessel functions of the first kind, denoted as $\mathrm{I_v(x)}$ are fundamental solutions to the modified Bessel differential equation;

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+v^2)y=0$ where v is the order real or complex x is real or complex argument

Series representation $v \in \mathrm{z}^{+}(non\ negative\ integer)$ $\mathrm{I_v(x)}$ admits a series expansion

$\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(v+k+1)}\left(\frac{x}{2}\right)^{v+2k}$ where $\Gamma(.)$ Is a gamma function

Recurrence relation

Order-Reduction Relation

$$\mathrm{I_{v-1}(x)-I_{v+1}(x)=\frac{2v}{x}I_v(x)} \tag{4.3}$$

This relates the difference of the neighbors to the function itself, scaled by $\frac{\mathrm{2v}}{\mathrm{x}}$

The Derivative Relation

$$\mathrm{\frac{d}{dx}I_v(x)=\frac{1}{2}(I_{v-1}(x)+I_{v+1}(x))} \tag{4.4}$$

This states that the derivative of $\mathrm{I_v(x)}$ is the average of its two neighboring functions.

4.4.2. Laplace Transform

Definition.4.1 Dyke.P.P, (2001) Laplace transform is defined as $\mathbf{F(t)}$ is a real or complex-valued function of the (time) variable $t\ >\ 0$ and s is a real or complex parameter

$f(s)=\int_0^{\infty}F(t)e^{-st}\,dt$ The notation $\mathrm{L\{F(t)\}}$ is used to denote the Laplace transform of the function $\mathrm{F(t)}$.

Properties of Laplace transform

1. Linearity If $F_1(\mathbf{t})$ and $F_2(\mathbf{t})$ are two functions whose Laplace transform exists, then $\mathrm{L\{a}F_1(\mathbf{t})+\mathrm{b}F_2(\mathbf{t}))\}=\mathrm{aL}\{F_1(\mathbf{t})\}+\mathrm{b}L\{F_2(\mathbf{t})\}$ where a and b are arbitrary constants.

2. Shifting property; if $\mathrm{L\{F(t)\}=f(s)}$ then $\mathrm{L\{e^{at}F(t)\}=f(s-a)}$

3. Differentiation ; if $\mathrm{L\{F(t)\}=f(s)}$ then $\mathrm{L\{F'(t)\}}=sf(s)-F(0)$

Inverse Laplace transform. If we are given a function f we can find its Laplace transform by evaluating the corresponding integral: $\mathrm{L\{F(t)\}=f(s)}$ It is also possible to go in the opposite direction: We are given $\mathrm{f}(s)$ and asked to find a function $\mathrm{F}(\mathrm{t})$, for which $\mathrm{f} = \mathrm{L}^{-1}\{\mathrm{F}\}$, that means the inverse Laplace transform.

5. A FLUID QUEUE DRIVEN BY SINGLE SERVER MARKOVIAN QUEUE WITH VARIANT WORKING VACATIONS

5.1. Model Description and Governing Equations

We consider an M/M/1 queue with variant working vacations as a background process. Customers arrive according to a Poisson process with rate λ .The service is provided by a single server with service rate μ. and it is exponentially distributed. If there are no customers in the system, then the server begins a vacation of random length which is exponentially distributed with parameter η . If the server finds customer at a vacation completion instant then it returns to a regular busy period; otherwise, it takes another vacation and continues to have K vacations sequentially. After K vacations the server switches to regular busy period. During these K vacations the server renders service generally with a slower rate and the service time follows exponential distribution with the parameter η. This type of vacation policy is referred as variant working vacation (VWV). If $\mathrm{K} \longrightarrow \infty$ then the queueing model becomes multiple working vacation model and $\mathrm{K} = 1$ gives the results for single working vacation.

Let $\mathrm{L(t)}$ be the number of customers at a time t and $\mathrm{J(t)}$ be the status of the server at a time t which is defined as follows

$$\mathrm{J(t)} = \begin{cases} j \text{ the server is on its } (j+1)^{th} \text{ working vacation at a time } t \text{ for} \\ \qquad j = 0, 1, 2, 3 \ldots k-1 \\ K \text{ the server is idle or busy at a time } t \end{cases}$$

The process $\{(\mathrm{L(t)}, \mathrm{J(t)}) : \mathrm{t} \geq 0\}$ defines is a continuous Markov process with state space

$$\Omega = \{(\mathrm{n}, \mathrm{j}) : \mathrm{n} \geq 0, \mathrm{j} = 0,1,2 \ldots \mathrm{K}\}$$

Let $\mathrm{C(t)}$ be the content of the buffer at time t. It is assumed that the content of the buffer increases at the rate of r_i *where* $i = 1,2,3$ when there are customers in the background queueing model, while the buffer content decreases at the rate r_0 when the system is empty. The dynamics of the buffer content process is given by

$$\frac{dC(t)}{dt} = \begin{cases} 0\ if\ \left(L(t), J(t)\right) = (0,0), C(t) = 0 \\ r_{0,} if\ \left(L(t), J(t)\right) = (0,0), C(t) > 0 \\ r_1, if\ \left(L(t), J(t)\right) = (n, 0), n \geq 1 \\ r_2, if\ \left(L(t), J(t)\right) = (n, 1), n \geq 1 \\ r_3, if\ \left(L(t), J(t)\right) = (n, 2), n \geq 1 \end{cases}$$

The 3D dimensional process $\{\left(\mathrm{L(t), J(t), C(t)}\right); \mathrm{t} > 0\}$ represents the fluid queue driven by M/M/1 queue with variant working vacation subject to the stability condition given by

$$\rho = \frac{\lambda}{\mu} < 1, and\ d^{\wedge} = r_0 P_{00} + r_1 \sum_{n=1}^{\infty} P_{n0} + r_2 \sum_{n=1}^{\infty} P_{n1} + r_3 \sum_{n=1}^{\infty} P_{n2} < 0$$

Where, P_{nj} represents the steady-state probability of the background queueing model to be in state (n, j)

The background process $\{\left(\mathrm{L(t), J(t)}\right);\ \mathrm{t} \geq 0\}$ is stable if and only if $\rho < 1$. The quantity d is called the mean drift of the process $\{\mathrm{C(t)};\ \mathrm{t} \geq 0\}$. When the buffer is infinite, the stochastic process $\{\left(\mathrm{L(t), J(t), C(t)}\right);\ \mathrm{t} \geq 0\}$ is stable if the mean drift $d^\wedge < 0$ and $\rho < 1$.

Symbols and descriptions

Table 1 Symbols and descriptions

Symbols	**Descriptions**
L(t)	The number of customers at a time t
J(t)	The status of the server at a time t
C(t)	The content of the buffer at time t
$F(u)$	the stationary probability distribution of the buffer content *C*
$F^{\wedge}(s)$	The Laplace transform of $F(u)$
$F^{*}(s)$	The Laplace stieltijies transform
P_{nj}	Steady state probabilities of the background queuing model to be in state (n, j)
λ	Arrival rate
η	Service rate during vacation
μ	Service rate in busy period
θ	Vacation interruption time
ρ	Traffic intensity
a	Empty buffer content
$d^{\wedge}$	Mean drift rate

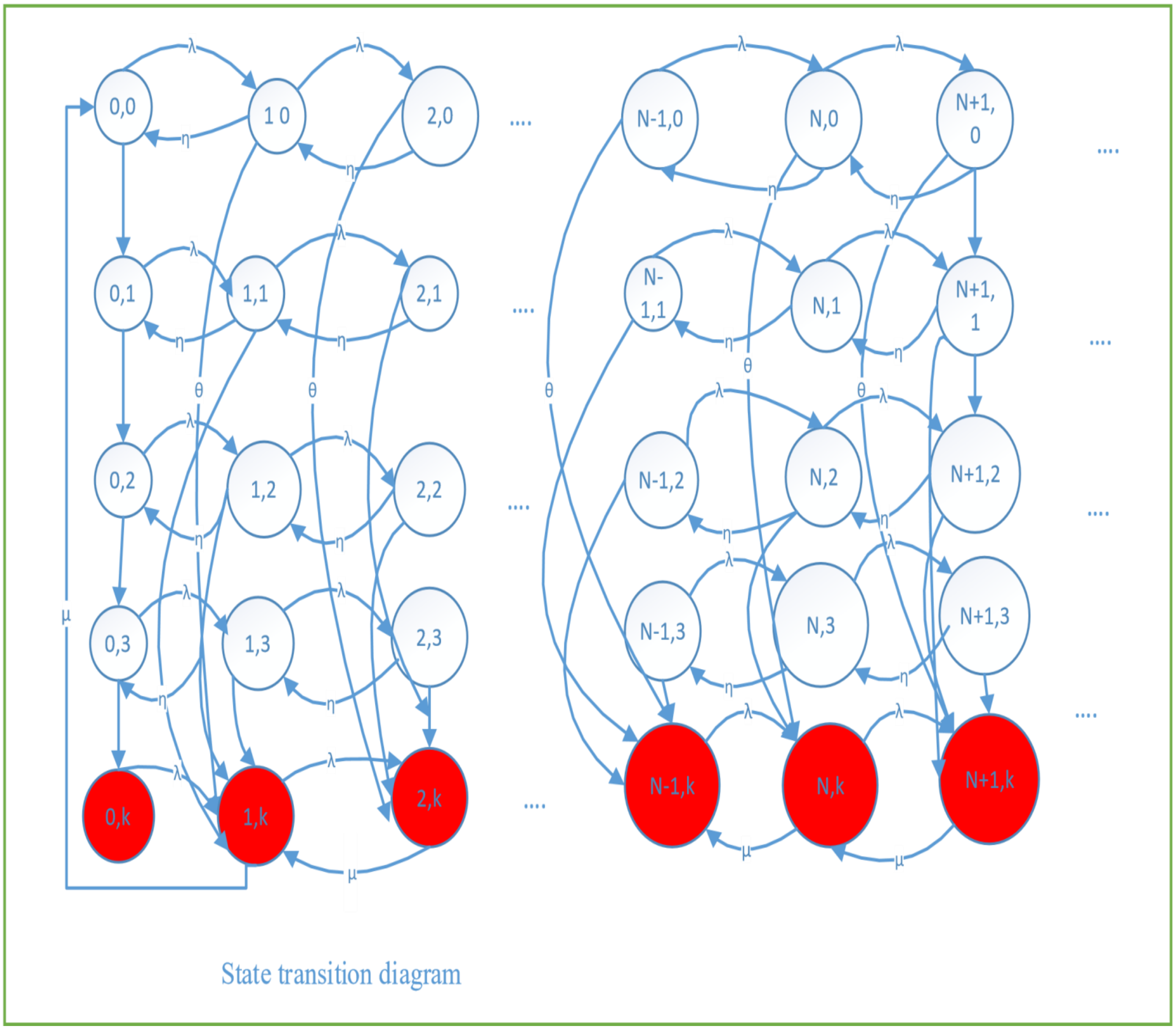

Figure 2 State Transition Diagram

5.2. Assumptions of the Model

The model includes the following assumptions

1. The inter-arrival time has a Poisson probability distribution with mean arrival rate λ customer arrivals per unit time
2. The service time has an exponential probability distribution with mean service rate μ service completion per unit time
3. The service discipline is First in First out (FIFO)
4. Infinite buffer

5. Arrival rate is less than service rate $\lambda < \mu$
6. The content of the buffer increases at the rate of r when there are customers in the background queueing model, while the buffer content decreases at the rate r_0 when the system is empty

Background Process of the M/M/1 Queuing System

From state transition diagram we can get the following steady state equations

$$(\lambda + \theta)P_{00} = \mu P_{1k} + \eta P_{10} \tag{1}$$

$$(\lambda + \eta + \theta)P_{10} = \lambda P_{00} + \eta P_{20} \tag{2}$$

$$(\lambda + \eta + \theta)P_{n0} = \lambda P_{n-1\,0} + \eta P_{n+1\,0}\,, n \geq 2 \tag{3}$$

$$(\lambda + \theta)P_{01} = \theta P_{00} + \eta P_{11} \tag{4}$$

$$(\lambda + \eta + \theta)P_{11} = \lambda P_{01} + \eta P_{21} \tag{5}$$

$$(\lambda + \eta + \theta)P_{nj} = \lambda P_{n-1\,j} + \eta P_{n+1\,j} \ n \geq 2 \ 1 \leq j \ \leq k - 1 \tag{6}$$

$$\lambda P_{0\,k} = \theta P_{0\,k-1} \tag{7}$$

$$(\lambda + \mu)P_{1k} = \lambda P_{0k} + \theta \sum_{j=0}^{k-1} P_{1j} + \mu P_{2j} \tag{8}$$

$$(\lambda + \mu)P_{nk} = \lambda P_{n-1\,k} + \theta \sum_{j=0}^{k-1} P_{nj} + \mu P_{n+1\,k}\,, n \geq 2 \tag{9}$$

$$\sum_{n=0}^{\infty} \sum_{j=0}^{k} P_{n,j} = 1 \tag{10}$$

From these equations we can get steady state probabilities of background process

By using probability generating function $G_0(z) = \sum_{n=0}^{\infty} P_{n0} z^n$

Multiplying equation (1),(2),(3) by z^n for n and summing over all possible values of n, we get

$$(\lambda + \theta) \sum_{n=0}^{\infty} P_{n0} z^n + \eta \sum_{n=1}^{\infty} P_{n0} z^n = \mu P_{1k} + \lambda \sum_{n=1}^{\infty} P_{n-1\,0}\, z^n + \eta \sum_{n=0}^{\infty} P_{n+1\,0}\, z^n$$

This implies

$$(\lambda + \theta) \sum_{n=0}^{\infty} P_{n0} z^n + \eta \sum_{n=1}^{\infty} P_{n0} z^n = \mu P_{(1,k)} + \lambda z \sum_{n=0}^{\infty} P_{n0}\, z^n + \eta z^{-1} \sum_{n=1}^{\infty} P_{n0}\, z^n$$

$$\lambda \sum_{n=0}^{\infty} P_{n0} z^n - \lambda z \sum_{n=0}^{\infty} P_{n0} z^n + \eta \sum_{n=1}^{\infty} P_{n0} z^n - \eta z^{-1} \sum_{n=1}^{\infty} P_{n0} z^n + \theta \sum_{n=1}^{\infty} P_{n0} z^n = \mu P_{1k}$$

Finally, rearranging the terms

$$(\lambda - \lambda z)\sum_{n=0}^{\infty} P_{n0} z^n + (\eta \ - \eta z^{-1})\sum_{n=1}^{\infty} P_{n0} z^n + \theta \sum_{n=0}^{\infty} P_{n0} z^n = \mu P_{1k}$$

$$\lambda(1-z)\mathrm{G}_0(\mathrm{z}) + \eta\ (1-z^{-1})(\mathrm{G}_0(\mathrm{z}) - \mathrm{P}_{00}) + \theta \mathrm{G}_0(\mathrm{z}) = \mu P_{1k}$$

$$\lambda(1-z)\mathrm{G}_0(\mathrm{z}) + \eta\ (1-z^{-1})(\mathrm{G}_0(\mathrm{z}) - \eta\ (1-z^{-1})\mathrm{P}_{00} + \theta \mathrm{G}_0(\mathrm{z}) = \mu P_{(1,k)}$$

$$[\lambda(1-z) + \eta\ (1-z^{-1}) + \theta]\mathrm{G}_0(\mathrm{z}) = \eta\ (1-z^{-1})\mathrm{P}_{00} + \mu P_{1k}$$

Simplifying this we obtain $\mathrm{G}_0(\mathrm{z}) = \frac{\mu P_{(1,k)} + \eta\ (1-z^{-1})\mathrm{P}_{00}}{[\lambda(1-z)+\eta\ (1-z^{-1})+\theta]}$ (11)

Again for $\mathrm{j} = 1,2 \ldots \mathrm{k}-1$ using generating function $\mathrm{G_j}(\mathrm{z}) = \sum_{j=1}^{k-1}\sum_{n=0}^{\infty} P_{nj}\, Z^n$

$$(\lambda + \eta + \theta)\sum_{j=1}^{k-1}\sum_{n=0}^{\infty} P_{nj}\, Z^n = \theta \mathrm{P}_{00} + \ \lambda \sum_{j=1}^{k-1}\sum_{n=0}^{\infty} P_{n-1\ j}\, Z^n + \eta \sum_{j=1}^{k-1}\sum_{n=0}^{\infty} P_{n+1\ j}\, Z^n$$

this implies

$$(\lambda + \eta + \theta)\sum_{j=1}^{k-1} G_j(z) = \theta \mathrm{P}_{00} + \ \lambda \sum_{j=1}^{k-1}\sum_{n=0}^{\infty} P_{nj}\, Z^n + \eta z^{-1} \sum_{j=1}^{k-1}\sum_{n=1}^{\infty} P_{nj}\, Z^n$$

Then $(\lambda + \eta)\sum_{j=1}^{k-1} G_j(z) + \theta \sum_{j=1}^{k-1} G_j(z) = \theta \mathrm{P}_{00} + \ \lambda \mathrm{z} \sum_{j=1}^{k-1}\sum_{n=0}^{\infty} P_{nj}\, Z^n +$
$\eta z^{-1} \sum_{j=1}^{k-1}\sum_{n=1}^{\infty} P_{(n,j)}\, Z^n$

$$(\lambda(1-\mathrm{z}))\sum_{j=1}^{k-1} G_j(z) + \eta(1-z^{-1})\sum_{j=1}^{k-1} G_j(z) + \theta \sum_{j=1}^{k-1} G_j(z) = \theta \mathrm{P}_{00} - \ \eta z^{-1} \sum_{j=1}^{k-1} P_{0j}$$

Finally we get

$$[\lambda(1-\mathrm{z}) + \eta(1-z^{-1}) + \theta]\sum_{j=1}^{k-1} G_j(z) = \theta \mathrm{P}_{00} - \ \eta z^{-1} \sum_{j=1}^{k-1} P_{0j}$$

$$\sum_{j=1}^{k-1} G_j(z) = \frac{\left[\theta \mathrm{P}_{00} - \eta z^{-1} \sum_{j=1}^{k-1} P_{0j}\right]}{[\lambda(1-\mathrm{z}) + \eta(1-z^{-1}) + \theta]} \qquad (12)$$

Again for j=k,

$\lambda \sum_{n=0}^{\infty} P_{nk}\, z^n + \mu \sum_{n=0}^{\infty} P_{nk}\, z^n = \theta P_{0\ k-1} + \lambda \sum_{j=1}^{\infty} P_{n-1\ k}\, z^n + \mu \sum_{j=1}^{\infty} P_{n+1\ k}\, z^n +$
$\theta \sum_{j=1}^{k-1}\sum_{n=1}^{\infty} P_{n+1\ k}\, z^n$

this implies

$\lambda \sum_{n=0}^{\infty} P_{nk}\, z^{n} + \mu \sum_{n=0}^{\infty} P_{nk}\, z^{n} = \theta P_{0\,k-1} + \lambda z \sum_{n=0}^{\infty} P_{nk}\, z^{n} + \mu z^{-1} \sum_{n-2}^{\infty} P_{nk}\, z^{n} +$
$\theta \sum_{j=1}^{k-1} \sum_{n=1}^{\infty} P_{nj}\, z^{n}$

$\lambda(1-z) \sum_{n=0}^{\infty} P_{nk}\, z^{n} + \mu \sum_{n=2}^{\infty} P_{nk}\, z^{n} + \mu z P_{1k} = \theta P_{0\,k-1} + \mu z^{-1} \sum_{n-2}^{\infty} P_{nk}\, z^{n} +$
$\theta \sum_{j=1}^{k-1} \sum_{n=1}^{\infty} P_{nj}\, z^{n}$
$\lambda(1-z) G_K(Z) + \mu \sum_{n=0}^{\infty} P_{nk}\, z^{n} - \mu z^{-1} \sum_{n-0}^{\infty} P_{nk}\, z^{n} - \mu P_{0k} + \mu z^{-1} P_{0k} + \mu P_{1k} =$
$\theta P_{0\,k-1} + \theta \sum_{j=1}^{k-1} G_J\,(Z) - \theta \sum_{J=1}^{K-1} P_{0j}$
$[\lambda(1-z) + \mu(1-z^{-1})] G_K(Z) - \mu P_{0k} + \mu z^{-1} P_{0k} + \mu P_{1k} = \theta P_{0\,k-1} + \theta \sum_{j=1}^{k-1} G_J\,(Z) -$
$\theta \sum_{J=1}^{K-1} P_{0j}$

$$[\lambda(1-z) + \mu(1-z^{-1})] G_K(Z) - \mu(1-z^{-1}) P_{0k} + \mu P_{1k} = \theta\left[P_{0\,k-1} - \sum_{J=1}^{K-1} P_{0j}\right] + \theta \sum_{j=1}^{k-1} G_J\,(Z) \qquad (13)$$

From equation (11) we have $\mathrm{G}_0(\mathrm{z}) = \frac{\mu P_{(1,k)} + \eta\,(1-z^{-1})\mathrm{P}_{00}}{[\lambda(1-z) + \eta\,(1-z^{-1}) + \theta]}$

Since $\mathrm{G}_0(1) = \sum_{n=0}^{\infty} P_{(n,0)}$ and $\mathrm{z} = 1$ is the root of the denominator right hand side of the equation (11)

We have that $\mathrm{z} = 1$ must be the root of the numerator of the right-hand side of the equation (11)

$$\theta \mathrm{G}_0(1) = \mu P_{1k} \qquad (14)$$

Nothing that $\mathrm{G}_0(1) = \mathrm{P}_{00}$

$$\theta \mathrm{P}_{00} = \mu P_{1k}$$

$$P_{1k} = \frac{\theta}{\mu}\ \mathrm{P}_{00} \qquad (15)$$

From this $\mathrm{G}_0(\mathrm{z}) = \frac{\theta \mathrm{P}_{00} + \eta\,(1-z^{-1})\mathrm{P}_{00}}{[\lambda(1-z) + \eta\,(1-z^{-1}) + \theta]}$

$$\mathrm{G}_0(\mathrm{z}) = \frac{[\theta + \eta\,(1-z^{-1})]}{[\lambda(1-z) + \eta\,(1-z^{-1}) + \theta]}\ P_{00} \qquad (16)$$

Since $G_j(1) = \sum_{n=0}^{\infty} P_{nj} < \infty$ and $\mathrm{z} = 1$ is the root of the denominator of the equation (12)

Then

$$\sum_{j=1}^{k-1} G_j(z) = \frac{\left[\theta \mathrm{P}_{00} - \eta z^{-1} \sum_{j=1}^{k-1} P_{0j}\right]}{[\lambda(1-\mathrm{z}) + \eta(1-z^{-1}) + \theta]}$$

$$\theta \sum_{j=1}^{k-1} G_j(1) = \theta \mathrm{P}_{00} - \eta \sum_{j=1}^{k-1} P_{0j}$$

$$\sum_{j=1}^{k-1} P_{0j} = \frac{\theta\left(1-\theta^{j-1}\right)}{\eta}\ \mathrm{P}_{00}$$

Finally, equation (12) becomes

$$\textstyle\sum_{j=1}^{k-1} G_j(z) = \frac{\left[\theta \mathrm{P}_{00} - \eta z^{-1} \frac{\theta\left(1-\theta^{j-1}\right)}{\eta}\ \mathrm{P}_{00}\right]}{[\lambda(1-\mathrm{z})+\eta(1-z^{-1})+\theta]}$$

$$\textstyle\sum_{j=1}^{k-1} G_j(z) = \frac{[\theta - \theta\left(1-\theta^{j-1}\right) z^{-1}\ ]\mathrm{P}_{00}}{[\lambda(1-\mathrm{z})+\eta(1-z^{-1})+\theta]} \tag{17}$$

From equation (13) we have

$[\lambda(1-z)+\mu(1-z^{-1})]G_K(Z) - \mu(1-z^{-1})P_{0k} + \mu P_{1k} = \theta[P_{0\,k-1} - \sum_{J=1}^{K-1} P_{0j}] + \theta \sum_{j\,=1}^{k-1} G_j\,(Z)$

Using equations (7) and (15)

$$P_{0k}\ = \frac{\theta}{\lambda}\ P_{0\,k-1}$$

$W G_K(Z) - \mu(1-z^{-1})\frac{\theta}{\lambda} \sum_{J=1}^{K-1} P_{0j}\ + \theta\ \ \mathrm{P}_{00} = \theta P_{0\,k-1} - \theta \sum_{J=1}^{K-1} P_{0j} + \theta \sum_{j\,=1}^{k-1} G_j\,(Z)$

where $W = [\lambda(1-z)+\mu(1-z^{-1})]$

This is same as

$W G_K(Z) - \mu(1-z^{-1})\frac{\theta}{\lambda} \sum_{J=1}^{K-2} P_{0j}\ + \theta\ \ \mathrm{P}_{00} = \theta \sum_{J=1}^{K-2} P_{0j} - \theta \sum_{J=1}^{K-1} P_{0j} + \theta \sum_{j\,=1}^{k-1} G_j\,(Z)$

$$W G_K(Z) = [\ \left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}\ -\theta\ \ +\theta^{j-2} - \theta^{j-1}]\mathrm{P}_{00} + \theta \textstyle\sum_{j\,=1}^{k-1} G_j\,(Z) \tag{18}$$

substituting equation (17) in equation (18) we get

$$W G_K(Z) = [\ \left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}\ -\theta\ \ +\theta^{j-2} - \theta^{j-1}]\mathrm{P}_{00} + \theta \sum_{j\,=1}^{k-1} G_j\,(Z)$$

$$\textstyle\sum_{j=1}^{k-1} G_j(z) = \frac{[\theta - \theta\left(1-\theta^{j-1}\right) z^{-1}\ ]\mathrm{P}_{00}}{[\lambda(1-\mathrm{z})+\eta(1-z^{-1})+\theta]}$$

$$G_K(Z) = \frac{\left[\left[\ \left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}\ -\theta\ \ +\theta^{j-2} - \theta^{j-1}\right]\mathrm{P}_{00} + \theta \sum_{j\,=1}^{k-1} G_j(Z)\right]}{w}$$

$$G_K(Z) = \frac{\left[\left[\ \left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}\ -\theta\ \ +\theta^{j-2} - \theta^{j-1}\right]\mathrm{P}_{00} + \theta\,\frac{[\theta - \theta\left(1-\theta^{j-1}\right) z^{-1}\ ]\mathrm{P}_{00}}{[\lambda(1-\mathrm{z})+\eta(1-z^{-1})+\theta]}\right]}{w}$$

$$G_K(Z)=\frac{\Bigg[\Big[\left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}-\theta\ +\theta^{j-2}-\theta^{j-1}\Big]+\theta\frac{[\theta-\theta(1-\theta^{j-1})z^{-1}]}{[\lambda(1-z)+\eta(1-z^{-1})+\theta]}\Bigg]P_{00}}{w} \tag{19}$$

Next we drive the equation for P_{00} by using equations (10), (11) ,(17) and (19)

$\sum_{n=0}^{\infty}\sum_{j=0}^{k}P_{(n,j)}=1$

$$G_0(z)+\sum_{n=0}^{\infty}\sum_{j=1}^{k-1}P_{nj}+G_K(z)=1$$

$$\frac{\mu P_{(1,k)}+\eta\,(1-z^{-1})P_{00}}{[\lambda(1-z)+\eta\,(1-z^{-1})+\theta]}+\frac{[\theta-\ \theta(1-\theta^{j-1})z^{-1}\,]}{[\lambda(1-z)+\eta(1-z^{-1})+\theta]}P_{00}$$
$$+\frac{\Bigg[\Big[\left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}-\theta\ +\theta^{j-2}-\theta^{j-1}\Big]+\theta\frac{[\theta-\ \theta(1-\theta^{j-1})z^{-1}\,]}{[\lambda(1-z)+\eta(1-z^{-1})+\theta]}\Bigg]P_{00}}{w}=1$$

$$P_{00}$$
$$=\frac{\theta+\eta\,(1-z^{-1})}{[\lambda(1-z)+\eta\,(1-z^{-1})+\theta]}+\frac{[\theta-\ \theta(1-\theta^{j-1})z^{-1}\,]}{[\lambda(1-z)+\eta(1-z^{-1})+\theta]}$$
$$+\frac{\Bigg[\Big[\left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}-\theta\ +\theta^{j-2}-\theta^{j-1}\Big]+\theta\frac{[\theta-\ \theta(1-\theta^{j-1})z^{-1}\,]}{[\lambda(1-z)+\eta(1-z^{-1})+\theta]}\Bigg]^{-1}}{w}$$

$$P_{00}=\left[\frac{\theta+\eta\,(1-z^{-1})+[\theta-\ \theta(1-\theta^{j-1})z^{-1}\,]}{L}+\frac{\left(\mu(1-z^{-1})\frac{\theta}{\lambda}\right)^{j-2}-\theta\ +\theta^{j-2}-\theta^{j-1}}{W}+\theta\frac{(\theta-\ \theta(1-\theta^{j-1})z^{-1})}{WL}\right]^{-1} \tag{20}$$

Where $L=\lambda(1-z)+\eta\,(1-z^{-1})+\theta$

$W=[\lambda(1-z)+\mu(1-z^{-1})]$

5.3. Stationary Analysis of the Fluid Queue

Define the joint probability distribution functions of the Markov process $\{(L(t),J(t),C(t));$ $t\geq 0\}$ at time *t* is

$F_{00}(t,u)\ =\ Pr\{(L(t)=0,J(t)=0,C(t))\ \leq\ u\}$

$F_{\ nj}(t,u)=Pr\{L(t)=n,J(t)=J,C(t)\leq\ u\}$, and $n=0,1,2,\ldots,\ j\ =\ 0,1,2,3$

When the process $\{(L(t), J(t), C(t)), t \geq 0\}$ is stable, its stationary random vector is denoted by (L, J, C).

Under steady-state conditions,

Let $F_{nj}(u) = \lim_{t\to\infty} \Pr\{L(t) = n, J(t) = j, C(t) \leq u\}$,

$=Pr\{L = n, J = j, C \leq u\}\, for\, u > 0$ and $(\mathrm{n}, \mathrm{j}) \in \Omega$

Where, The variable u is a real number that represents the threshold for the continuous component of the state space and Ω is state space

Then, the stationary probability distribution of the buffer content *C* is given by

$$F(u) = P\{C \leq u\} = F_{00}(u) + \sum_{n=1}^{\infty} \sum_{j=0}^{k} F_{nj}\,(u)$$

Note that the probability for the content of the buffer to be empty is given by

$$P\{C = 0\} = F_{00}(0) + \textstyle\sum_{n=1}^{\infty} \sum_{j=0}^{2} F_{nj}\,(0) \tag{21}$$

By standard arguments, the system of differential difference equations that governs the fluid queueing model is given by

$$r_0 \frac{d}{du} F_{00}(u) = \mu F_{13}(u) - (\lambda + \theta) F_{00}(u) + \eta F_{10}(u) \tag{22}$$

$$\mathrm{r}_1 \frac{d}{du} F_{(n,0)}(u) = \lambda F_{(n-1,0)}(u) - (\lambda + \eta + \theta) F_{(n,0)}(u) + \eta F_{(n+1,0)}(u)\ n \geq 1 \tag{23}$$

$$r_0 \frac{d}{du} F_{01}(u) = \theta F_{00}(u) - (\lambda + \eta + \theta) F_{01}(u) + \eta F_{11}(u) \tag{24}$$

$$\mathrm{r}_1 \frac{\mathrm{d}}{\mathrm{du}} \mathrm{F}_{\mathrm{n}1}(\mathrm{u}) = \lambda \mathrm{F}_{\mathrm{n}-1\,1}(\mathrm{u}) - (\lambda + \eta + \theta) \mathrm{F}_{\mathrm{n}1}(\mathrm{u}) + \eta \mathrm{F}_{\mathrm{n}+1\,1}(\mathrm{u})\ \mathrm{n} \geq 1 \tag{25}$$

$$\mathrm{r}_2 \frac{\mathrm{d}}{\mathrm{du}} \mathrm{F}_{\mathrm{n}2}(\mathrm{u}) = \lambda \mathrm{F}_{\mathrm{n}-1\,2}(\mathrm{u}) - (\lambda + \eta + \theta) \mathrm{F}_{\mathrm{n}2}(\mathrm{u}) + \eta \mathrm{F}_{\mathrm{n}+1\,2}(\mathrm{u})\ \mathrm{n} \geq 1 \tag{26}$$

$$r_0 \tfrac{d}{du} F_{03}(u) = \theta F_{02}(u) - \lambda F_{03}(u) \tag{27}$$

$$\mathrm{r}_3 \tfrac{d}{du} F_{n3}(u) = \lambda F_{n-1\,3}(u) - (\lambda + \mu) F_{n3}(u) + \theta \textstyle\sum_{j=0}^{2} F_{1j}(u) + \mu F_{n+1\,3}(u)\ n \geq 2 \tag{28}$$

subject to boundary condition

$$F_{00}(0) = a\,, F_{nj}(0) = 0, \lim_{n\to\infty} F_{nj}(u) = P_{nj}\ where\ \Omega\ =\ \{(\mathrm{n}, \mathrm{j}); n \neq 0\ and\ j \neq 0\} \tag{29}$$

The constant $a,$ such that $0 < a < 1$ needs to be determined. Since we make an assumption that the content of the buffer increases at the rate r when there are one or more customers in the background queueing model, it is impossible to have the buffer empty when the modulating process is in any of the state (n, j) for $n = 1, 2, 3, \ldots$ and $j = 0,1,2,3$ However, when the background queueing model is empty, the buffer content depletes at rate $r_0 < 0$ and hence, with some positive probability, it is possible that the content is empty. Therefore, the boundary conditions given by Equation (29) is valid.

To determine the constant $'a'$which represents $F_{00}(0),$ adding Equations $(23) - (29)$ yields

$$ro \frac{d}{du} F_{00}(u) + r_J \sum_{n=1}^{\infty} \sum_{j=0}^{3} F_{nj}\ (u) = 0 \tag{30}$$

Integrating from zero to infinity and using boundary conditions

$$ro(\lim_{u \to \infty} F_{00}(u) - F_{00}(0)) + r_j(\sum_{n=1}^{\infty} \sum_{j=0}^{3} \lim_{u \to \infty} F_{nj}(u) - \sum_{n=1}^{\infty} \sum_{j=0}^{3} F_{nj}(0)) = 0 \tag{31}$$

Note that

$$F_{nj}(\infty) = \lim_{t \to \infty} \Pr\{L(t) = n, J(t) = j\ C(t) \leq \infty\}$$

$$= \lim_{t \to \infty} \Pr\{L(t) = n, J(t) = j\ \} = P_{nj}\ \ (n, j) \in \Omega$$

Using boundary conditions represented in equation (29) in equation (31) we obtain

$$r_0 F_{00}(0) = r_0 \left(\lim_{u \to \infty} F_{00}(u)\right) + r_1 \sum_{n=1}^{\infty} \lim_{u \to \infty} F_{n0}(u) + r_2 \sum_{n=1}^{\infty} \lim_{u \to \infty} F_{n1}(u) +$$

$$r_3 \sum_{n=1}^{\infty} \lim_{u \to \infty} F_{n2}(u)$$

which on simplification yields

$$F_{00}(0) = \frac{r_0 P_{00} + r_J (\sum_{n=1}^{\infty} \sum_{j=0}^{3} P_{nj})}{ro} = \frac{d}{ro} \tag{32}$$

Taking Laplace transform of equation (24)

$$\mathrm{r}_1 \frac{d}{du} F_{n0}(u) = \lambda F_{n-1\,0}(u) - (\lambda + \eta + \theta) F_{n0}(u) + \eta F_{n+1\,0}(u)\ n \geq 2$$

$$\mathrm{r}_1 s F^{\wedge}{}_{n0}(s) = \lambda F^{\wedge}{}_{n-1\,0}(s) - (\lambda + \eta + \theta) F^{\wedge}{}_{n0}(s) + \eta F^{\wedge}{}_{n+1\,0}(s)$$

$$(\mathrm{r}_1 \mathrm{s} + \lambda + \eta + \theta) F^{\wedge}{}_{n\,0}(s) - \eta F^{\wedge}{}_{n+1\,0}(s) = \lambda F^{\wedge}{}_{n-1\,0}(s)$$

Which can be written as

$$\frac{F^\wedge{}_{n\,0}(s)}{F^\wedge{}_{n-1\,0}(s)} = \frac{\lambda}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-\frac{\eta F^\wedge{}_{n+1\,0}(s)}{F^\wedge{}_{n\,0}(s)}}$$

Recursively it follows that

$$\frac{F^\wedge{}_{n\,0}(s)}{F^\wedge{}_{n-1\,0}(s)} = \frac{\lambda}{(\mathrm{rs}+\lambda+\eta+\theta)-\frac{\lambda\eta}{(\mathrm{rs}+\lambda+\eta+\theta)-\frac{\lambda}{(\mathrm{rs}+\lambda+\eta+\theta)-\frac{\eta F^\wedge{}_{n+2\,0}(s)}{F^\wedge{}_{n+2\,0}(s)}}}}$$

The above continued fraction is represented using the following notation

$$\frac{F^\wedge{}_{n\,0}(s)}{F^\wedge{}_{n-1\,0}(s)} = \frac{\lambda}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-}\,\frac{\lambda\eta}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-}\,\frac{\lambda\eta}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-}\cdots$$

Now assume $f(s) = \frac{\lambda\eta}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-}\,\frac{\lambda\eta}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-}\cdots$

Then

$$\frac{F^\wedge{}_{n\,0}(s)}{F^\wedge{}_{n-1\,0}(s)} = \frac{f(s)}{\eta}$$

And hence

$$F^\wedge{}_{n\,0}(s) = \frac{f(s)}{\eta}F^\wedge{}_{n-1\,0}(s) = (\frac{f(s)}{\eta})^n\,F^\wedge{}_{0\,0}(s) \tag{34}$$

Also $f(s)$ can be rewritten as

$$f(s) = \frac{\lambda\eta}{(\mathrm{r}_1\mathrm{s}+\lambda+\eta+\theta)-f(s)} = \frac{\frac{\lambda\eta}{r}}{\left(\mathrm{s}+\frac{\lambda+\eta+\theta}{r_1}\right)-\frac{f(s)}{r_1}}$$

which leads quadratic equation given by

$$\frac{f(s)^2}{r_1} - \left(\mathrm{s}+\frac{\lambda+\eta+\theta}{r_1}\right)f(s) + \frac{\lambda\eta}{r_1} = 0$$

upon solving the above equation, we get

$$f(s) = \frac{p2-\sqrt{p2^2-\alpha 2^2}}{\frac{2}{r_1}}\ ,p_2 = \mathrm{s}+\frac{\lambda+\eta+\theta}{r_1},\alpha_2 = \frac{2\sqrt{\lambda\eta}}{r_1}$$

$$F^\wedge{}_{n\,0}(s) = \left(\frac{f(s)}{\eta}\right)^n\,F^\wedge{}_{0\,0}(s)$$

$$= \left[\frac{p2-\sqrt{{p_2}^2-{\alpha_2}^2}}{\frac{2\eta}{r_1}}\right]^n F^{\wedge}{}_{0\,0}(s)$$

$F^{\wedge}{}_{n\,0}(s) = (\frac{r_1}{2\eta})^n\ [p2-\sqrt{{p_2}^2-{\alpha_2}^2}]^n F^{\wedge}{}_{0\,0}(s)$ (35)

which on inversion yields

$F_{n0}(u) = (\frac{r_1}{2\eta})^n k I_k(\alpha_2 u)\alpha_2^k \exp[\left(-\frac{\lambda+\eta+\theta}{r}\right)u] * F_{00}(u)$ (36)

Taking the Laplace transforms of the equation (22) - (28) we get

$(r_0 s+\lambda+\theta)\hat{F}_{00}(s) = \mu\hat{F}_{13}(s)+\eta\hat{F}_{10}(s)+r_0 a$ (22.1)

$(r_1 s+\lambda+\eta+\theta)\hat{F}_{n0}(s) = \lambda\hat{F}_{n-1,0}(s)+\eta\hat{F}_{n+1,0}(s)$ (23.1)

$(r_0 s+\lambda+\eta+\theta)\hat{F}_{01}(s) = \theta\hat{F}_{00}(s)+\eta\hat{F}_{11}(s)$ (24.1)

$(r_1 s+\lambda+\eta+\theta)\hat{F}_{n1}(s) = \lambda\hat{F}_{n-1,1}(s)+\eta\hat{F}_{n+1,1}(s)$ (25.1)

$(r_2 s+\lambda+\eta+\theta)\hat{F}_{n2}(s) = \lambda\hat{F}_{n-1,2}(s)+\eta\hat{F}_{n+1,2}(s)$ (26.1)

$(r_0 s+\lambda)\hat{F}_{03}(s) = \theta\hat{F}_{02}(s)$ (27.1)

$(r_3 s+\lambda+\mu)\hat{F}_{n3}(s) = \lambda\hat{F}_{n-1,3}(s)+\mu\hat{F}_{n+1,3}(s)+\theta\sum_{j=1}^{2}F_{1j}(s)$ (28.1)

Using a recurrence relation with constant coefficients for equation (23.1)

$$(r_1 s+\lambda+\eta+\theta)\hat{F}_{n0}(s) = \lambda\hat{F}_{n-1,0}(s)+\eta\hat{F}_{n+1,0}(s)$$

The solution for $\hat{F}_{n0}(s)$ is written as $\hat{F}_{n0}(s) = A\alpha_1^n + B\alpha_2^n$ α_1 *and* α_2 are the roots of the characteristics equation $\eta\alpha^2-(r_1 s+\lambda+\eta+\theta)\alpha+\lambda = 0$.

$$\alpha = \frac{(r_1 s+\lambda+\eta+\theta)\pm\sqrt{(r_1 s+\lambda+\eta+\theta)^2-4\eta\lambda}}{2\eta}$$

let α_1 be stable root $/\alpha_1/< 1$ α_2 be unstable root $/\alpha_2/\geq 1$

$\hat{F}_{n0}(s) = C\alpha_1^n$ for n=1 $\hat{F}_{10}(s) = C\alpha_1$ and $\hat{F}_{00}(s) = C$

Thus $\hat{F}_{n0}(s) = \hat{F}_{00}(s)\alpha_1^n$ (37)

In order to write $\hat{F}_{n1}(s)$ in terms of $\hat{F}_{00}(s)$ we use second order recurrence relation

$\eta\hat{F}_{n+1,1}(s)-(r_1 s+\lambda+\eta+\theta)\hat{F}_{n,1}(s)+\lambda\hat{F}_{n-1,1}(s) = 0$

This can be written as

$\eta\boldsymbol{\beta^2}-(r_1 s+\lambda+\eta+\theta)\beta+\lambda = 0$ now

$\beta = \frac{(r_1 s+\lambda+\eta+\theta)\pm\sqrt{(r_1 s+\lambda+\eta+\theta)^2-4\eta}\lambda}{2\eta}$ and the general solution is assumed to be

$\hat{F}_{n1}(s) = C\boldsymbol{\beta}^{\boldsymbol{n}}$, $\hat{F}_{11}(s) = C\boldsymbol{\beta}$, $\hat{F}_{01}(s) = C$ relate $\hat{F}_{01}(s)$ to $\hat{F}_{00}(s)$

$$(r_0 s + \lambda + \eta + \theta)\hat{F}_{01}(s) = \theta\hat{F}_{00}(s) + \eta\hat{F}_{11}(s)$$

$$(r_0 s + \lambda + \eta + \theta)C = \theta\hat{F}_{00}(s) + \eta C\boldsymbol{\beta}$$

$\boldsymbol{C} = \frac{\theta\hat{F}_{00}(s)}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}$ then the general solution for $\hat{F}_{n1}(s)$ becomes

$$\hat{F}_{n1}(s) = \frac{\theta\hat{F}_{00}(s)}{(r_0 s + \lambda + \eta + \theta) - \eta\boldsymbol{\beta}}\boldsymbol{\beta}^{\boldsymbol{n}} \tag{38}$$

Again using equation (25.1) applying 2^{nd} order recurrence relation we compute $\hat{F}_{n2}(s)$ as

$\eta\boldsymbol{\Upsilon}^{\boldsymbol{2}} - (r_2 s + \lambda + \eta + \theta)\Upsilon + \lambda = 0$

$$\Upsilon = \frac{(r_2 s + \lambda + \eta + \theta) \pm \sqrt{(r_2 s + \lambda + \eta + \theta)^2 - 4\eta}\lambda}{2\eta}$$

The general solution for $\hat{F}_{n2}(s)$ is written as $\hat{F}_{n2}(s) = C\Upsilon^{\mathrm{n}}$ $\hat{F}_{02}(s) = C$

$\hat{F}_{n2}(s) = \hat{F}_{02}(s)\Upsilon^{\mathrm{n}}$ from boundary conditions and $\hat{F}_{n1}(s) = \frac{\theta\hat{F}_{00}(s)}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\boldsymbol{\beta}^{\boldsymbol{n}}$

for $\mathrm{n} = 0$ $\hat{F}_{01}(s) = \frac{\theta\hat{F}_{00}(s)}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}$

$$\hat{F}_{n2}(s) = \left(\frac{\theta}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\right)^{\boldsymbol{2}}\boldsymbol{\Upsilon}^{\boldsymbol{n}}\hat{F}_{00}(s) \tag{39}$$

Finally using equation (27.1) and (28.1) we can solve for $\hat{F}_{n,3}(s)$

$$(r_3 s + \lambda + \mu)\hat{F}_{n,3}(s) = \lambda\hat{F}_{n-1,3}(s) + \mu\hat{F}_{n+1,3}(s) + \theta\sum_{j=0}^{2} F_{1j}(s) \tag{40}$$

Rearranging and using 2^{nd} order non- homogeneous recurrence relation equation (40) can be written as

$\mu\hat{F}_{n+1,3}(s) - (r_3 s + \lambda + \mu)\hat{F}_{n,3}(s) + \lambda\hat{F}_{n-1,3}(s) = \theta\sum_{j=0}^{2} F_{1j}(s)$

$\mu d^2 - (r_3 s + \lambda + \mu)d + \lambda = \theta\sum_{j=0}^{2} F_{1j}(s)$ first solve for homogenous part

$\mu d^2 - (r_3 s + \lambda + \mu)d = 0$ the general solution for homogenous part is given as

$\hat{F}_{n,3}(s) = Cd^n$ $d = \frac{(r_3 s+\lambda+\mu)\pm\sqrt{(-(r_3 s+\lambda+\mu)^2-4\mu}}{2\mu}$ due to non-homogenous term $\theta\sum_{j=0}^{2} F_{1j}(s)$

we need to relate these to $\hat{F}_{00}(s)$

$\theta(F^{\wedge}{}_{10}(s) + F_{11}(s) + F_{12}(s)) = \theta[\alpha_1 + \frac{\theta\beta}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}} + \left(\frac{\theta}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\right)^2 \Upsilon]\hat{F}_{00}(s)$

$\theta\sum_{j=0}^{2}F_{1j}(s)=\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]F_{00}^{\wedge}(s)$ Where $B=(r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}$

Particular solution

$$(r_3s+\lambda+\mu)F_{n,3}^{\wedge part}(s)=\lambda F_{n-1,3}^{\wedge part}(s)+\mu F_{n+1,3}^{\wedge part}(s)+\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]F_{00}^{\wedge}(s)$$

Assume particular solution is independent of n

$$F_{n,3}^{\wedge part}(s)=\frac{\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]F_{00}^{\wedge}(s)}{(r_3s+\lambda+\mu-\lambda-\mu}$$

Particular solution is not constant instead assume a form proportional to $F_{00}^{\wedge}(s)$

$$F_{n,3}^{\wedge part}(s)=KF_{00}^{\wedge}(s)$$

Substituting back we get

$$K=\frac{\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]}{r_3s}$$

Then $F_{n,3}^{\wedge}(s)=Cd^n+\frac{\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]}{r_3s}$

to determine C use boundary conditions at $\mathrm{n}=0$

$F_{0,3}^{\wedge}(s)=C+\frac{\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]}{r_3s}$ again from equation (28.1)

$F_{03}^{\wedge}(s)=\frac{\theta}{(r_0s+\lambda)}F_{02}^{\wedge}(s)=\frac{\theta}{(r_0s+\lambda)}\left(\frac{\theta}{\boldsymbol{B}}\right)^2F_{00}^{\wedge}(s)$

Thus $C=[\frac{\theta^3}{(r_0s+\lambda)B^2}-\frac{\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]}{r_3s}]\,F_{00}^{\wedge}(s)$

Combining all terms the general solution for $F_{n3}^{\wedge}(s)$ is

$F_{n3}^{\wedge}(s)=Cd^n+\frac{\theta[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon]}{r_3s}F_{00}^{\wedge}(s)$

$F_{n3}^{\wedge}(s)=\left[\frac{\theta^3}{(r_0s+\lambda)B^2}-\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}\right]d^nF_{00}^{\wedge}(s)+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}$

$$F_{n3}^{\wedge}(s)=\left[\frac{\theta^3}{(r_0s+\lambda)B^2}d^n+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}(1-d^n)\right]F_{0,0}^{\wedge}(s)$$

where $B = (r_0 s + \lambda + \eta + \theta) - \eta\boldsymbol{\beta}$ (41)

From equation (22.1)

$$(r_0 s + \lambda + \theta)F^{\wedge}_{00}(s) = \mu F^{\wedge}_{13}(s) + \eta F^{\wedge}_{10}(s) + r_0 a$$

$$F^{\wedge}_{00}(s) = \frac{\mu}{r_0}\frac{1}{(s+\frac{\lambda+\theta}{r_0})}F^{\wedge}_{13}(s) + \frac{\eta}{r_0}\frac{1}{(s+\frac{\lambda+\theta}{r_0})}F^{\wedge}_{10}(s) + \frac{a}{s+\frac{\lambda+\theta}{r_0}} \tag{42}$$

Using equation (37) and (41) for n=1

$$F^{\wedge}_{1,3}(s) = \left[\frac{\theta^3}{(r_0 s+\lambda)B^2}d + \frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3 s}(1-d)\right]F^{\wedge}_{0,0}(s) \tag{43}$$

$$F^{\wedge}_{10}(s) = F^{\wedge}_{00}(s)\alpha_1 \tag{44}$$

Substituting (43) and (44) in (42) we get

$$F^{\wedge}_{00}(s) = \frac{\mu}{r_0}\frac{1}{(s+\frac{\lambda+\theta}{r_0})}\left[\frac{\theta^3}{(r_0 s+\lambda)B^2}d + \frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3 s}(1-d)\right]F^{\wedge}_{0,0}(s) + \frac{\eta}{r_0}\frac{1}{(s+\frac{\lambda+\theta}{r_0})}F^{\wedge}_{00}(s)\alpha_1 + \frac{a}{s+\frac{\lambda+\theta}{r_0}}$$

$$F^{\wedge}_{00}(s) = \frac{a}{s+\frac{\lambda+\theta}{r_0} - \mu\frac{1}{\left(s+\frac{\lambda+\theta}{r_0}\right)}\left[\frac{\theta^3}{(r_0 s+\lambda)B^2}d + \frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3 s}(1-d)\right] - \eta\alpha_1} \tag{45}$$

5.4. Buffer Content Distribution

The stationary buffer content distribution of the fluid under consideration is given by

$$F(u) = P\{C \le u\} = F_{00}(u) + \sum_{n=1}^{\infty}\sum_{j=0}^{2}F_{nj}(u) + \sum_{n=0}^{\infty}F_{n3}(u)$$

$$F(u) = F_{00}(u) + \sum_{n=0}^{\infty}F_{n0}(u) + \sum_{n=0}^{\infty}F_{n1}(u) + \sum_{n=0}^{\infty}F_{n2}(u) + \sum_{n=0}^{\infty}F_{n3}(u) \tag{46}$$

Laplace transform of equation (46) gives

$$F^{\wedge}(s) = F^{\wedge}_{00}(s) + \sum_{n=0}^{\infty}F^{\wedge}_{n0}(s) + \sum_{n=0}^{\infty}F^{\wedge}_{n1}(s) + \sum_{n=0}^{\infty}F^{\wedge}_{n2}(s) + \sum_{n=0}^{\infty}F^{\wedge}_{n3}(s)$$

Substituting equations (37), (38), (39), (41) in (42) we get

$$F^{\wedge}(s) = \hat{F}_{00}(s) + \sum_{n=0}^{\infty} \hat{F}_{00}(s)\alpha_1^n + \sum_{n=0}^{\infty} \frac{\theta \hat{F}_{00}(s)}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}} \boldsymbol{\beta^n} +$$

$$\sum_{n=0}^{\infty} \left(\frac{\theta}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\right)^{\mathbf{2}} \boldsymbol{\Upsilon^n} \hat{F}_{00}(s) + \sum_{0}^{\infty} \left[\frac{\theta^3}{(r_0 s+\lambda)B^2} d^n + \frac{\theta\left[\alpha_1 + \frac{\theta\beta}{\boldsymbol{B}} + \left(\frac{\theta}{\boldsymbol{B}}\right)^2 \Upsilon\right]}{r_3 s}(1-d^n)\right] \hat{F}_{00}(s) \quad (47)$$

$$F^{\wedge}(s) = \left[1 + \frac{1}{1-\alpha_1} + \frac{\theta}{(r_0 s + \lambda + \eta + \theta) - \eta\boldsymbol{\beta})(\mathbf{1}-\boldsymbol{\beta})}\right.$$

$$+ \left(\frac{\theta^2}{((r_0 s + \lambda + \eta + \theta) - \eta\boldsymbol{\beta})^{\mathbf{2}}(\mathbf{1}-\Upsilon)}\right) + \frac{\theta^3}{(r_0 s + \lambda)B^2(1-d)}$$

$$\left. + \frac{\theta\left[\alpha_1 + \frac{\theta\beta}{\boldsymbol{B}} + \left(\frac{\theta}{\boldsymbol{B}}\right)^2 \Upsilon\right]}{r_3 s}\left(N + 1 - \frac{1-d^N+1}{1-d}\right)\right] \hat{F}_{00}(s)$$

Where

$$\hat{F}_{00}(s) = \frac{a}{s + \frac{\lambda+\theta}{r_0} - \mu \frac{1}{\left(s + \frac{\lambda+\theta}{r_0}\right)}\left[\frac{\theta^3}{(r_0 s + \lambda)B^2} d + \frac{\theta\left[\alpha_1 + \frac{\theta\beta}{\boldsymbol{B}} + \left(\frac{\theta}{\boldsymbol{B}}\right)^2 \Upsilon\right]}{r_3 s}(1-d)\right] - \eta\alpha_1}$$

After Substituting $\hat{F}_{00}(s)$ in (47) upon simplification leads to $F^{\wedge}(s) =$

$$a\left[\frac{\left[1+\frac{1}{1-\alpha_1}+\frac{\theta}{(r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta})(\mathbf{1}-\boldsymbol{\beta})}+\left(\frac{\theta^2}{((r_0 s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta})\text{^}\mathbf{2}(\mathbf{1}-\Upsilon)}\right)+\frac{\theta^3}{(r_0 s+\lambda)B^2}(1-d)+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3 s}\left(N+1-\frac{1-d^N+1}{1-d}\right)\right]}{s+\frac{\lambda+\theta}{r_0}-\mu\frac{1}{\left(s+\frac{\lambda+\theta}{r_0}\right)}\left[\frac{\theta^3}{(r_0 s+\lambda)B^2}d+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3 s}(1-d)\right]-\eta\alpha_1}\right]$$

(48)

The inverse Laplace transform of (47)

$$F(u)=\left[\delta(u)+\sqrt{\frac{\eta}{\lambda}}\frac{e^{-\frac{\lambda+\eta+\theta}{r_1}u}}{r_1}I_1\left(\frac{2\sqrt{\eta\lambda}}{r_1}u\right)+\frac{\theta}{\sqrt{r_0r_1}}e^{-\frac{\lambda+\eta+\theta}{r_1}u}I_0\left(2\sqrt{\frac{\eta\lambda}{r_0r_1}}u\right)+\right.$$

$$\frac{\theta^2}{\sqrt{r_0r_1r_2}}e^{-\frac{(\lambda+\eta+\theta)}{r_2}u}I_1\left(2\sqrt{\frac{\eta\lambda}{r_2}}u\right)+\frac{\theta^3}{r_0}e^{-\frac{\lambda}{r_0}u}*\left[\frac{u}{r_0}e^{-\frac{(\lambda+\eta+\theta)}{r_0}u}I_1\left(2\sqrt{\frac{\eta\lambda}{r_0}}u\right)\right]*$$

$$\left.\left[\frac{1}{\mu}e^{-\frac{(\lambda+\mu)}{r_3}u}I_0\left(2\sqrt{\frac{\mu\lambda}{r_3}}u\right)\right]\right]*F_{00}(u)$$

Here, denote the Laplace stieltjies transform (LSTs) of the stationary joint distribution of the fluid model and the stationary distribution of the buffer content as

$$F_{nj}^{\wedge}(s)=\int_0^{\infty}e^{-su}dF_{nj}(u)$$

$$F^{*}(s)=\int_0^{\infty}e^{-su}dF\,(u)$$

$$F^{*}(s)=sF^{\wedge}(s)$$

$$F^{*}(s)=sF_{00}^{\wedge}(s)+s\sum_{n=0}^{\infty}F_{00}^{\wedge}(s)\alpha_1^n+s\sum_{n=0}^{\infty}\frac{\theta F_{00}^{\wedge}(s)}{(r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\boldsymbol{\beta}^{\boldsymbol{n}}+$$

$$s\sum_{n=0}^{\infty}\left(\frac{\theta}{(r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\right)^{\mathbf{2}}\boldsymbol{\Upsilon}^{\boldsymbol{n}}F_{00}^{\wedge}(s)+s\sum_0^{\infty}\left[\frac{\theta^3}{(r_0s+\lambda)B^2}d^n+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}(1-d^n)\right]F_{00}^{\wedge}(s)$$

After simplification

$$F^{*}(s)=s\left[1+\frac{1}{1-\alpha_1}+\frac{\theta}{(B)(\mathbf{1}-\boldsymbol{\beta})}+\frac{\theta^2}{B^{\mathbf{2}}(\mathbf{1}-\Upsilon)}+\frac{\theta^3}{(r_0s+\lambda)B^2}(1-d)+\right.$$

$$\left.\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}\left(-\frac{1}{1-d}\right)\right]F_{00}^{\wedge}(s)$$

(49)

With the normalization condition $\lim_{s\to 0}F^{*}(s)=1$ we can determine the expression of probability 'a ' of the empty buffer content

$$\lim_{s\to 0}F^{\wedge}(s)=\frac{\theta\left[\alpha_1+\frac{\theta\beta_0}{\boldsymbol{B_0}}+\left(\frac{\theta}{\boldsymbol{B_0}}\right)^2\beta_0\right]\left(-\frac{1}{1-d_0}\right)}{r_3}\left[\frac{a}{\frac{\lambda+\theta}{r_0}-\eta\alpha_1}\right]$$

$$\frac{\theta\left[\alpha_1+\frac{\theta\beta_0}{\boldsymbol{B_0}}+\left(\frac{\theta}{\boldsymbol{B_0}}\right)^2\beta_0\right]\left(-\frac{1}{1-d_0}\right)\left[\frac{a}{\frac{\lambda+\theta}{r_0}-\eta\alpha_1}\right]}{r_3}=1$$

$$a=\frac{-r_3(1-d_0)[\frac{\lambda+\theta}{r_0}-\eta\alpha_1]}{\theta\left[\alpha_1+\frac{\theta\beta_0}{\boldsymbol{B_0}}+\left(\frac{\theta}{\boldsymbol{B_0}}\right)^2\beta_0\right]}$$ ′ a ′ is empty buffer content distribution

5.5. Some Performance Measures in Fluid Model

Mean buffer content

The average amount of fluid stored in the buffer in steady state. This is a direct analogue to the mean number of customers in a traditional queue.

Using equation (49) and simplifying we get

$$F^{\wedge}(s)=\hat{F}_{00}(s)+\sum_{n=0}^{\infty}\hat{F}_{00}(s)\alpha_1^n+\sum_{n=0}^{\infty}\frac{\theta\hat{F}_{00}(s)}{(r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\boldsymbol{\beta^n}+$$
$$\sum_{n=0}^{\infty}\left(\frac{\theta}{(r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta}}\right)^{\mathbf{2}}\boldsymbol{\Upsilon^n}\hat{F}_{00}(s)+\left[\frac{\theta^3}{(r_0s+\lambda)B^2}d^n+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}(1-d^n)\right]\hat{F}_{00}(s)$$

after simplification we get

$$F^{\wedge}(s)=\left[1+\frac{1}{1-\alpha_1}+\frac{\theta}{(r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta})(\mathbf{1}-\boldsymbol{\beta})}+\left(\frac{\theta^2}{((r_0s+\lambda+\eta+\theta)-\eta\boldsymbol{\beta})\text{^}\mathbf{2}(\mathbf{1}-\Upsilon)}\right)+\frac{\theta^3}{(r_0s+\lambda)B^2}(1-d)+\right.$$
$$\left.\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}\left(N+1-\frac{1-d^N+1}{1-d}\right)\right]\hat{F}_{00}(s) \tag{50}$$

Applying Laplace stieltijies transform

$$F^{*}(s)=s\left[1+\frac{1}{1-\alpha_1}+\frac{\theta}{(B)(\mathbf{1}-\boldsymbol{\beta})}+\frac{\theta^2}{B^{\mathbf{2}}(\mathbf{1}-\Upsilon)}+\frac{\theta^3}{(r_0s+\lambda)B^2}(1-d)\right.$$
$$\left.+\frac{\theta\left[\alpha_1+\frac{\theta\beta}{\boldsymbol{B}}+\left(\frac{\theta}{\boldsymbol{B}}\right)^2\Upsilon\right]}{r_3s}\left(-\frac{1}{1-d}\right)\right]\hat{F}_{00}(s)$$

The mean buffer content distribution is given as

$$E(C) = \frac{dF^*(s)}{ds} \text{ at } s = 0$$

$$E(C) = \left[\frac{2\eta}{\eta-\lambda+\theta-\sqrt{(\lambda+\eta+\theta)^2-4\eta\lambda}} + 8\eta\left(\frac{r_0\theta(-\eta+\lambda+\theta)+r_1(\lambda+\eta+\theta)}{(\lambda+\eta+\theta)^2(\eta-\lambda-\theta)^2}\right) + \frac{\theta^2(-2\left(r_0-\eta\beta_0'\right)(1-\Upsilon_0)+B_0\Upsilon_0'}{B_0^3(1-\Upsilon)^2} + \theta^3\left(\frac{1-d(0)}{\lambda B(0)^2}\right)\right]$$

Server utilization

The probability that buffer is non-empty is obtained as

Utilization $=1-\sum_{j=0}^{N} F_j(0)$

$$= 1 - F_0(0) = 1 - a$$

6. NUMERICAL RESULTS AND DISCUSSIONS

In this chapter, some numerical results were presented in the form of tables and graphs to demonstrate how change in parameters of the system influence its empty buffer content and stationary buffer content

6.1. Buffer Content Distribution

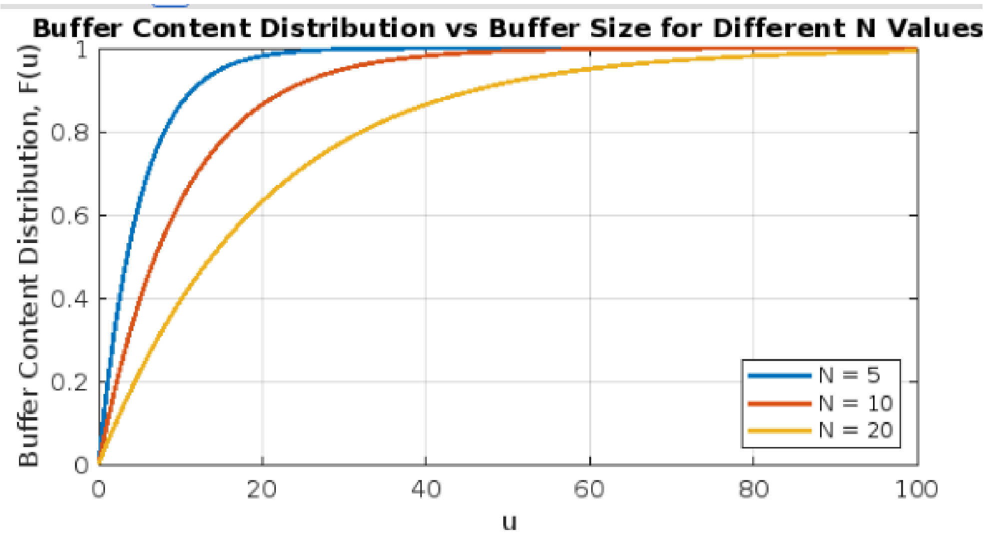

Figure 3 Buffer content distribution vs fluid level (u)

From figure 3, by assuming the parameter values for $\lambda = 1, \theta = 0.5, \mu = 2\ , \eta = 1.1\ r_0 = -2.5, r_1 = 1, r_2 = 1.5, r_3 = 1.75$ the behavior of buffer content F(u) against u for the same set of parameters varying values of θ observe that both the stability conditions $\lambda < \mu$ and $d < 0$,for this choice parameter of values the boundary conditions are given by $F_{00}(0) = a$ =0.090931 and $F_{n0}(0) = 0$ as U tends to infinity $F_{n0}(u)$ will converge to the corresponding to the steady state probabilities of background Queueing model P_{n0} therefore $F_{00}(0)$ starts with a =0.090931 and converges to $P_{00} = 0.5794$ as u increases, similarly $F_{n0}(u)$ starts with zero and increases with increase in U and converges to P_{n0} as U tends to infinity. F(u) is increasing with decrease in the parameter value theta (θ) and converges to 1 as U tends to Infinity.

6.2. Impact of Parameters on Empty Buffer Content

By assuming parameter values for $\lambda = 1, \theta = 0.5, \mu = 2\ , \eta = 1.1\ r_0 = -2.5, r_1 = 1, r_2 = 1.5, r_3 = 1.75$

From figure we depict that For a fixed $\theta\ and\ \eta$ the probability $'\mathrm{a}'$ the probability of empty buffer is increasing with increasing of service rate during busy period μ and this indicates with high service rate during busy period may lead to high probability of the buffer being empty.

Table 2 Variation Empty Buffer Content (‘a’) with service rate (μ)

μ	**a**
0.46154	0.030772
0.4761	0.05698
0.4878	0.07804
0.49485	0.09073
0.5	0.1
0.50394	0.10709

from the table we depict that for a fixed $\theta\ and\ \mu$ with increasing of the value of η service rate during working vacation increases empty buffer content ($'\mathrm{a}'$) probability of empty buffer content

Table 3 Variation Of Empty Buffer Content (‘a’) with service rate in working vacation (η)

η	**a**
0.5	0.1
0.50246	0.10443
0.50467	0.10841
0.50667	0.11201
0.50847	0.11525
0.51012	0.11822

Effect of vacation interruption time $(\boldsymbol{\theta})$ on buffer content distribution $\boldsymbol{F(u)}$

From figure 4 we depict that for a fixed values of fluid level (u), as θ increases the buffer content accumulation increases .This graph shows how the probability that the buffer content is below a specific level u changes as we vary θ

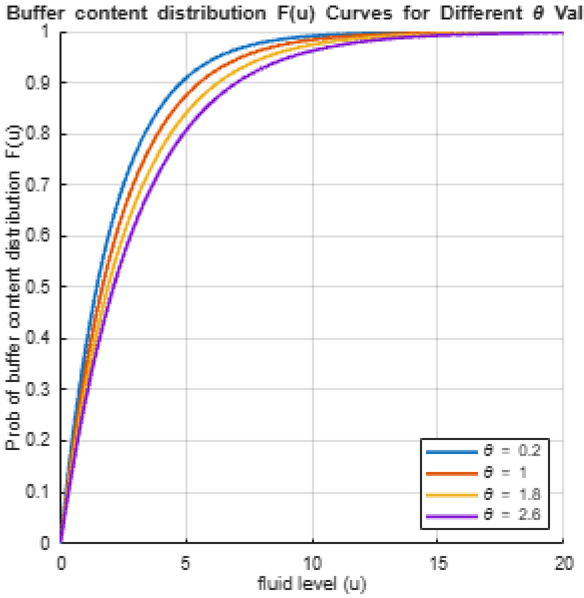

Figure 4 Effect of (vacation interruption time) θ on buffer content distribution F(u)

Effect of service rate during working vacation period and busy period on mean buffer content

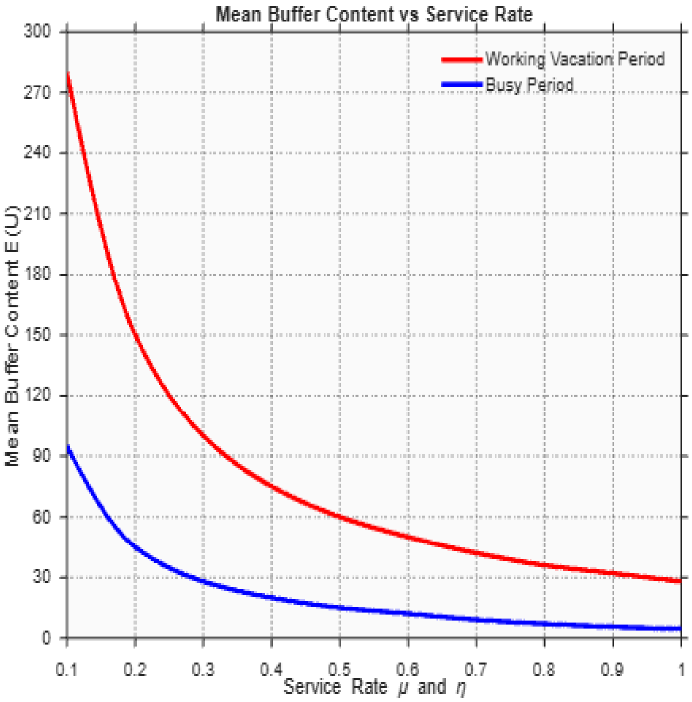

Figure 5 Effect of service rate on mean buffer content E(C)

From the figure 5, Demonstrate that changes in service rate directly affect the drain rate from the buffer. For a fluid queue with service rate μ during busy periods and reduced rate η during vacations. During busy periods with constant service rate μ, the mean buffer content depends on both the arrival rate and service rate. For an $\mathrm{M/M/1}$ driven fluid queue, the mean buffer content $E(C)$ during busy periods is inversely related to the service rate μ. Higher service rates reduce the average buffer content faster, while lower service rates lead to accumulation. Higher service rates during working vacations η reduce the rate of buffer accumulation.

Generally, some impact parameters on buffer content distribution and empty buffer distribution have been investigated numerically.

7. SUMMARY, CONCLUSION AND RECOMMENDATION

7.1. Summary

In this study fluid queue driven by Single Server Markovian queue with variant working vacations were analyzed. It assumes where the server takes up k sequential working vacations (reduced service rate) before resuming normal service. The expressions for steady state probabilities of background queuing model were obtained using probability generating functions(PGF) and the probabilities of fluid queue were obtained by using continued fraction, Laplace transform and modified Bessel function of first kind.

Derives closed-form expressions for steady-state probabilities and buffer content distribution.

Identifies stability conditions: arrival rate $\lambda < \mu$ service rate and negative mean drift $d < 0$

Demonstrates that buffer content increases with lower θ (vacation interruption rate) and converges to 1 as u tends to infinity. Models high-speed networks, production systems, and telecommunication traffic where servers experience intermittent reduced-speed phases.

7.2. Conclusion

The method of probability generating functions (PGF) was successfully used to derive the steady-state probabilities of the background Markovian queuing model, which includes the variant working vacation policy. The state transitions between working and busy vacation periods were solved in a manageable way by this method. Analysis was done on the buffer content's stationary distribution. The solution for the probability density function was found by determining the basic differential equations controlling the fluid flow in each background process state. The main idea behind this solution was to solve the equations using the Laplace transform and obtain explicit expressions for the buffer content distribution after handling the interconnected nature of the equations across various states using recurrence relations.

7.3. Recommendations

Future research work should extend this model by incorporating non-Markovian arrivals (phase type distributions, M/G/1, GI/G/c) to relax exponential assumptions, while exploring multi-server or networked queuing systems to broaden applicability in domains like cloud computing.

Comparative studies with Bernoulli vacation or working breakdown models would quantify performance gains and diffusion approximations could be developed for large scale systems where exact analysis is intractable

Applications such as modelling sleep modes in 5G edge servers or traffic flows in smart cities should be investigated with focus on transient analysis for time varying workloads. Collaborations with industry partners could refine practical constraints and sensitivity analysis of key parameters would further harden the models strength for real world deployment

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